

$$\text{Let } f_a = \begin{cases} 0, & x \in (0, 1) \\ 0, & x \in [1, \infty) \end{cases}, \quad F_a = \begin{cases} 0, & x \in (0, 1) \\ x^{-1/a}, & x \in (1, \infty) \end{cases}$$

$$g_a = \begin{cases} (x(\log \frac{2}{x})^2)^{-\frac{1}{a}}, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases}, \quad G_a = \begin{cases} 0, & x \in (0, 2) \\ (x(\log x)^2)^{-\frac{1}{a}}, & x \in (2, \infty) \end{cases}$$

$$\text{Since } f_a \in L^1 \Leftrightarrow a > 1, \quad f_a \in L^p \Leftrightarrow p < a. \quad (1)$$

$$\text{Since } F_a \in L^1 \Leftrightarrow a < 1, \quad F_a \in L^p \Leftrightarrow p > a. \quad (2)$$

For g and G , we will show that $g_a \in L^1 \Leftrightarrow a > 1$ and $G_a \in L^1 \Leftrightarrow a \leq 1$.

$$\int_0^1 g_1(x) dx = \int_0^1 (\log \frac{2}{x})^{-1} dx = (\log 2)^{-1} \Rightarrow g_1 \in L^1.$$

$$g_1'(x) = \frac{2}{x^2(\log \frac{2}{x})^3} - \frac{1}{x^2(\log \frac{2}{x})^2} \begin{cases} > 0, & x > 2/e^2 \\ = 0, & x = 2/e^2 \\ < 0, & 0 < x < 2/e^2 \end{cases}$$

$\Rightarrow g_1(x)$ decreases on $(0, 2/e^2]$ and increases on $(2/e^2, 1)$. Denote $2/e^2$ by c .

$$\text{For } a > 1, \quad g_a(x) \leq g_1(x) \text{ for } x \in (0, c] \Rightarrow \int_0^c g_a(x) dx \leq \int_0^c g_1(x) dx < \infty$$

$$g_a(x) \leq g_a(1) = (\log 2)^{-2/a} < \infty \text{ for } x \in (c, 1) \Rightarrow \int_c^1 g_a(x) dx < \infty$$

$$\Rightarrow g_a(x) \in L^1$$

For $a < 1$, let $y = \frac{1}{x}$. Calculate $\lim_{y \rightarrow \infty} \frac{\log^2(2y)}{y^{1-a}}$.

Use L'Hospital's rule,

$$\lim_{y \rightarrow \infty} \frac{2}{(1-a)^2} \frac{y-1}{y^a} = \lim_{y \rightarrow \infty} \frac{2 \log(2y)}{(1-a)y^{1-a}} = \lim_{y \rightarrow \infty} \frac{(2 \log(2y))/y}{(1-a)y^{-a}} = \lim_{y \rightarrow \infty} \frac{\log^2(2y)}{y^{1-a}}$$

$$\text{Since } \lim_{y \rightarrow \infty} \frac{2}{(1-a)^2} \frac{y-1}{y^a} = \frac{2}{(1-a)^2} \lim_{y \rightarrow \infty} y^{a-1} = 0, \quad \lim_{y \rightarrow \infty} \frac{\log^2(2y)}{y^{1-a}} = 0$$

$$\text{Thus } \lim_{x \rightarrow 0} (x(\log \frac{2}{x})^2)^{-\frac{1}{a}} / \frac{1}{x} = \lim_{y \rightarrow \infty} (y^{1-a} / \log^2(2y))^{\frac{1}{a}} = \infty. \text{ Hence,}$$

$$\exists \varepsilon \in (0, 1) \text{ s.t. } \forall x \in (0, \varepsilon), \quad (x(\log \frac{2}{x})^2)^{-\frac{1}{a}} / \frac{1}{x} > 1.$$