

18.125

PSET 8

DUE 4/22

Yinghui Wang

35/35

✓ 1. a)

$$\phi(p) = \int_X |f|^p d\mu$$

$$\text{Since } |f|^p \leq \begin{cases} |f|^r, & |f| \leq 1 \\ |f|^s, & |f| \geq 1 \end{cases} \leq |f|^r + |f|^s,$$

and $\int_X |f|^r d\mu = \phi(r) < \infty$, $\int_X |f|^s d\mu = \phi(s) < \infty$,
 $\int_X (|f|^r + |f|^s) d\mu = \phi(r) + \phi(s) < \infty$. Therefore,
 $\phi(p) = \int_X |f|^p d\mu \leq \phi(r) + \phi(s) < \infty$. Thus $p \in E$.

b) If $r, s \in \text{Int}(E) = \text{interior of } E$, then $0 < \phi(r), \phi(s) < \infty$.

✓ $\forall p \in (r, s)$, $p = \lambda r + (1-\lambda)s$, $\lambda \in (0, 1)$, by Hölder's inequality,

$$\int_X |f|^p d\mu = \int_X |f|^{\lambda r} |f|^{(1-\lambda)s} d\mu \leq \left(\int_X |f|^r d\mu \right)^\lambda \left(\int_X |f|^s d\mu \right)^{1-\lambda} \quad (*)$$

Thus $\log(\phi(p)) \leq \lambda \log(\phi(r)) + (1-\lambda) \log(\phi(s))$, so $\log(\phi)$ is convex in $\text{Int}(E)$. Therefore ϕ is continuous in $\text{Int}(E)$.

For any $a \in E^0 = \text{boundary of } E$, there exist a decreasing or a increasing sequence $\{s_n\} \subset \text{Int}(E)$ with $\lim s_n = a$. WLOG, assume $\{s_n\}$ is decreasing. Then $|f(x)|^{s_n} \rightarrow |f(x)|^a$, $\forall x \in X$.

Since $|f(x)|^{s_n} \leq \max\{|f(x)|^a, |f(x)|^{s_1}\}$ and $|f|^a, |f|^{s_1} \in L^1(\mu)$,

by Lebesgue's Dominated Convergence Thm, $\lim \phi(s_n) = \phi(a)$.

Hence ϕ is continuous on E .

✓ c) E can be any connected subset of $(0, \infty)$, not necessarily open / closed and E can be a single point.

Let μ be the Lebesgue measure,