

25/8 ④ Let f be an upper semicontinuous function on compact metric space X . Then $\{f < \alpha\}$ is open $\forall \alpha \in \mathbb{R}$, and $\bigcup_{\alpha \in \mathbb{R}} \{f < \alpha\}$ is an open cover of X (as f is given to be real-valued). If $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that $\bigcup_{i=1}^n \{f < \alpha_i\}$ is a finite subcover, then if $\max \alpha_i = M$, then $X \subseteq \bigcup_{i=1}^n \{f < \alpha_i\} \subseteq \{f < M\}$ which implies ^{is is} that f is bounded. Let s denote the supremum of f over the domain X , and consider the sets $\{f \geq s - \frac{1}{n}\} \forall n \in \mathbb{N}$. Clearly, $\{f \geq s - \frac{1}{n}\} \searrow \{f \geq s\}$, but we have also that $\{f \geq s - \frac{1}{n}\}$ is closed and therefore compact $\forall n$. By the ~~theorem~~ corollary to theorem 2.36 in W. Rudin, Principles of Mathematical Analysis, page 38, $\bigcap_{n \in \mathbb{N}} \{f \geq s - \frac{1}{n}\}$ is nonempty — which implies that $\{f \geq s\}$ is nonempty, so f achieves its maximum, as desired.

⑤ $\forall \alpha \in \mathbb{R}$, $\{g < \alpha\}$, $\{g > \alpha\}$, $\{h < \alpha\}$, $\{h > \alpha\}$ are countable unions and/or countable intersections of the sets A_i for g and B_i for h , respectively, and the Borel ~~field~~ σ -algebra is closed under countable unions and intersections. It follows that $g^{-1}((\alpha, \beta)) = \{g > \alpha\} \cap \{g < \beta\}$, ~~Borel~~ $h^{-1}((\alpha, \beta)) = \{h > \alpha\} \cap \{h < \beta\}$ are Borel for all segments (α, β) ; since all open sets V may be written as a countable union of such segments, $g^{-1}(V)$ and $h^{-1}(V)$ are Borel for all open V . This implies that g and h are Borel because the Borel field is exactly the σ -algebra generated by all the open sets, as desired.