

see (*)

$h-g=0$ μ -almost everywhere. Finally, it is easily seen that g and h are Borel functions because $g = 1_{A_1}$ and $h = 1_{B_1}$ on A_1 and B_1 , respectively, and the Borel σ -algebra is closed under countable union and intersection.

In the general case, write $f = f^+ - f^-$, and choose Borel functions g_1 and h_1 with $g_1 \leq f^+ \leq h_1$ (and $g_1 = h_1$ μ -almost everywhere) and g_2 and h_2 with $g_2 \leq f^- \leq h_2$ (and $g_2 = h_2$ μ -almost everywhere). Let $g = g_1 - h_2$ and $h = h_1 - g_2$; then $g = h$ μ -almost everywhere and $g = g_1 - h_2 \leq f^+ - f^- = f \leq h_1 - g_2 = h$. It then suffices to prove that the sum of two Borel functions is again Borel. Let j and k be Borel functions.

~~Let $f(x) = j(x) + k(x)$. We follow the model of the proof of theorem 1.8 and prove a more general statement:~~

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prove a more general statement: If $\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous, and $\ell(x)$ is defined by $\ell(x) = \Phi(j(x), k(x))$, then $\ell: X \rightarrow \mathbb{R}$ is Borel. Write $\ell(x) = (j(x), k(x))$.

If R is an open rectangle in $\mathbb{R}^2$, with sides parallel to the coordinate axes, then R is the cartesian product of two segments I_1 and I_2 and $f^{-1}(R) = j^{-1}(I_1) \cap k^{-1}(I_2)$, which is Borel, since j and k are Borel functions. Every open set V in $\mathbb{R}^2$ is a countable union of such rectangles R_i , and since $f^{-1}(V) = f^{-1}(\bigcup_{i=1}^{\infty} R_i) = \bigcup_{i=1}^{\infty} f^{-1}(R_i)$, $f^{-1}(V)$ is Borel for all open V . Since the Borel σ -algebra is the σ -algebra generated by all the open sets, it follows that $f^{-1}(W)$ is Borel for all Borel W , and that indeed f is Borel. Now if W is open in $\mathbb{R}$, $\Phi^{-1}(W)$ is open in $\mathbb{R}^2$ because W is continuous; hence $\ell^{-1}(W)$ is Borel. Since the Borel σ -algebra is the σ -algebra generated by all the open sets, it follows that $\ell^{-1}(W)$ is Borel for all Borel W , which, in turn, implies that ℓ is Borel, as desired. Since it is easily seen that $\Phi(x, y) = x + y$ is continuous (because it is upper semicontinuous and lower semicontinuous), we may conclude that $j + k$ is Borel, *quod erat demonstrandum*.

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