

Countable additivity of  $\mu$  follows from the finite additivity of the cardinality on finite sets. Finally,  $K$  is the support of  $\mu$  because  $\mu(K)=1$ , and if  $H \subseteq K$ ,  $H \neq K$ , then  $H \cap K$  is a strict subset of  $K$ , so its cardinality is less than that of  $K$  and  $\mu(H) < 1$ , as desired.

(ii) It follows from exercise 25 of the second chapter of W. Rudin, Principles of Mathematical Analysis, page 45, that every complex set contains a countable dense subset. If  $K$  has infinite cardinality, let  $K'$  be such a subset, and let  $\{k'_n\}$ ,  $n \in \mathbb{N}$ , be the points of  $K'$ .  $\forall S \in \mathcal{B}$ , define  $\mu(S) = \sum_{n: k'_n \in S} \frac{1}{2^n}$ . Clearly since  $k'_n \notin \emptyset \forall n \in \mathbb{N}$ ,  $\mu(\emptyset) = 0$ . To establish the countable additivity of  $\mu$ , note that if  $\{S_n\} \subseteq \mathcal{B}$ ,  $n \in \mathbb{N}$ , and  $S_i \cap S_j = \emptyset$  if  $i \neq j$ , then  $\mu(\bigcup_{i=1}^{\infty} S_i) = \sum_{n: k'_n \in \bigcup_{i=1}^{\infty} S_i} \frac{1}{2^n} = \sum_{i=1}^{\infty} \sum_{n: k'_n \in S_i} \frac{1}{2^n} = \sum_{i=1}^{\infty} \mu(S_i)$ , as desired, so  $\mu$  is a measure.  $K$  is the support of  $\mu$  because  $\mu(K) = \sum_{n: k'_n \in K} \frac{1}{2^n} = \sum_{n \in \mathbb{N}} \frac{1}{2^n} = 1$ , and if  $H \subseteq K$ ,  $H \neq K$ , and  $H$  is compact, then  $\mu(H) < 1$  because  $H$  cannot contain  $K'$ . (Assume the contrary. If  $H \supseteq K'$ , then, as  $H$  is closed,  $H \supseteq \overline{K'}$ . However  $K'$  is dense in  $K$ , so every point of  $K$  is either a point in  $K'$  or a limit point of  $K'$ ; hence  $\overline{K'} \supseteq K$ , contradicting the definition of  $H$ ). This completes the proof.

25/25 (3) We follow the model of the proof of the Vitali-Carathéodory Theorem. Assume first that  $f \geq 0$  and  $f$  is not identically 0. Consider an increasing sequence of simple functions  $\{s_n\}$ ,  $n \in \mathbb{N}$ , such that  $s_n \uparrow f$ , and define  $t_1 = s_1$ ,  $t_{n+1} = s_{n+1} - s_n \forall n \in \mathbb{N}$  so that  $f = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{j=1}^n t_j = \sum_{j=1}^{\infty} t_j$ . Since each  $t_n$  is a linear combination of characteristic functions, we see that there are measurable sets  $E_i$ , not necessarily disjoint, and constants  $c_i > 0$   $f(x) = \sum_{i=1}^{\infty} c_i \mathbb{1}_{E_i}(x) \forall x \in \mathbb{R}^N$ . By Stroock Theorem 2.1.12, for each  $E_i$  we may choose  $A_i \in \mathcal{F}_\sigma$  and  $B_i \in \mathcal{G}_\delta$  such that  $A_i \subseteq E_i \subseteq B_i$  and  $\mu(B_i \setminus A_i) = 0$ , where  $\mu$  denotes the Lebesgue measure. Since  $\mathcal{F}_\sigma$  and  $\mathcal{G}_\delta$  are subsets of the Borel  $\sigma$ -algebra, it follows that  $A_i$  and  $B_i$  are Borel sets  $\forall i$ . Define  $g(x) = \sum_{i=1}^{\infty} c_i \mathbb{1}_{A_i}(x)$  and  $h(x) = \sum_{i=1}^{\infty} c_i \mathbb{1}_{B_i}(x)$ ; clearly,  $g(x) \leq f(x) \leq h(x) \forall x$  and  $h - g = \sum_{i=1}^{\infty} c_i (\mathbb{1}_{B_i}(x) - \mathbb{1}_{A_i}(x)) = \sum_{i=1}^{\infty} c_i \mathbb{1}_{B_i \setminus A_i}(x)$ , so  $\int_{\mathbb{R}^N} (h - g) d\mu = \sum_{i=1}^{\infty} c_i \mu(B_i \setminus A_i) = 0$ , meaning that  $\Rightarrow$