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David B Rush

18.125 - Problem Set VII

Let V be open with $V \supseteq K$.

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① Since $V \supseteq \bigcap_{\alpha} K_{\alpha}$, we see that $V^c \subseteq \bigcup_{\alpha} K_{\alpha}^c$. Since V is open, V^c is closed, so by theorem 2.4, V^c is compact. Since the K_{α} are all compact, by the corollary to theorem 2.5, they are all closed. Hence K_{α}^c is open $\forall \alpha$, meaning $\{K_{\alpha}^c\}$ is an open cover of V^c . Let $K_1^c, \dots, K_n^c$ be a finite subcover. As K_i^c has measure 0 for $1 \leq i \leq n$, $\mu(\bigcup_{i=1}^n K_i^c) \leq \sum_{i=1}^n \mu(K_i^c) = 0$, so $\bigcup_{i=1}^n K_i^c$ has measure 0, and is open, meaning $\bigcap_{i=1}^n K_i$ is compact and has measure 1. But $V^c \subseteq \bigcup_{i=1}^n K_i^c$ implies $V \supseteq \bigcap_{i=1}^n K_i$, as desired. It follows immediately that K^c is the largest open set with measure 0, because if U is open and $\mu(U) = 0$, then U^c is compact and $\mu(U^c) = 1$, meaning $K \subseteq U^c$. Then $K^c \supseteq U$. (Also $\mu(K^c) = 0$ because $\mu(K) = 1$; this follows because if $\mu(K) = 1 - \varepsilon$, $\varepsilon > 0$, then $\exists V$ open, $V \supseteq K$, $\mu(V) \leq 1 - \frac{\varepsilon}{2}$, in which case V could not contain a set of measure 1, contradicting what has been proven). Finally, to see that $H \subseteq K$, $H \neq K$, H compact implies $\mu(H) < 1$, we need note only that, by definition of K , $\mu(H) = 1$, H compact, implies $K \subseteq H$, so $H \subseteq K$, H compact, $\mu(H) = 1$ implies $H = K$, contradicting the assumption that $H \neq K$, as desired.

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②

~~Let $K \subseteq \mathbb{R}$ be compact, and define $\mu(K) = \mu(K \cap \mathbb{R})$. Since $\mu(K) = 1$, we have $\mu(K) = 1$. Let μ be the Lebesgue measure on $\mathbb{R}$. Then $\mu(K) = 1$ implies $K \subseteq \mathbb{R}$. Clearly, $\mu(K) = 1$ implies $K \subseteq \mathbb{R}$.~~