

Claim 4.3. *The space X is locally compact Hausdorff.*

Proof. The space X is metric, and therefore Hausdorff. $\checkmark$

As shown above, every point (x, y) has a neighborhood $\{x\} \times \mathbb{R}$ homeomorphic to $\mathbb{R}$ in the usual topology, which is locally compact. Hence, so is X . $\square$

Claim 4.4. *For any function $f \in C_c(X)$, the set $A_f = \{x \in \mathbb{R} : f(x, y) \neq 0 \text{ for some } y \in \mathbb{R}\}$ is finite.*

Proof. If $f \in C_c(X)$, then there is a compact set $C \subset X$ containing the support of f . Hence, $A_f \subset A_C$. By the lemma above, A_C is finite. Thus, so is A_f . $\square$

Define the functional Λ on $C_c(X)$ by

$$\Lambda(f) = \sum_{x \in A_f} \int_{-\infty}^{+\infty} f(x, y) dy$$

Let μ be the measure given by the Riesz Representation Theorem, and $E = \mathbb{R} \times \{0\} \subset X$.

Claim 4.5. *For any $x \in \mathbb{R}$, the restriction of μ to $\{x\} \times \mathbb{R}$ gives us the usual Lebesgue measure on $\mathbb{R}$.*

Proof. Riesz representation theorem says that the measure μ is the unique measure such that functions f compactly supported on $\{x\} \times \mathbb{R}$ have

$$\int_{-\infty}^{+\infty} f(x, y) dy = \int_{\{x\} \times \mathbb{R}} f d\mu$$

hence, it has to be the Lebesgue measure. $\square$

Claim 4.6. *For any compact subset $C \subset E$, we have $\mu(C) = 0$.*

Proof. As $C \subset E$, then, as sets, $C = A_C$. Hence, C is a finite set.

Let $U_\epsilon = C \times (-\epsilon, \epsilon)$. Hence,

$$\mu(C) \leq \mu(U) = \sum_{x \in C} \mu(-\epsilon, \epsilon) = 2\epsilon \#C$$

for any $\epsilon > 0$. Hence, $\mu(C) = 0$. $\checkmark$ $\square$

Claim 4.7. *The measure μ is such that $\mu(E) = +\infty$.*

Proof. It is enough to prove that every open set containing $U \supset E$ has measure $+\infty$. Indeed, then

$$\mu(E) = \inf_{E \subset U} \mu(U) = +\infty$$

Let $n_x = \mu(U \cap \{x\} \times \mathbb{R})$. Note that $U \cap (\{x\} \times \mathbb{R})$ is a non-empty open subset of $\mathbb{R}$, hence its Lebesgue measure n_x is not zero. For every $m \in \mathbb{N}$, let

$$F_m = \{x \in \mathbb{R} : n_x > 1/m\}$$

Note that $\mathbb{R} = \cup_{m \in \mathbb{N}} F_m$, because for every $x \in \mathbb{R}$, there is m such that $n_x > 1/m$. Hence, not all F_m can be finite, otherwise $\mathbb{R}$ would be countable. Pick m_0 such that F_{m_0} has infinitely many elements.

Hence, for every $N \in \mathbb{N}$, we can pick $x_1, x_2, \dots, x_N \in F_{m_0}$, and

$$\mu(U) \geq \mu(\cup_{i=1}^N (U \cap (\{x_i\} \times \mathbb{R}))) = \sum_{i=1}^N \mu(U \cap (\{x_i\} \times \mathbb{R})) = \sum_{i=1}^N n_{x_i} > N \times 1/m_0$$

As this is true for every $N \in \mathbb{N}$, we have that $\mu(U) = +\infty$, as we wanted to prove. $\square$