

Exercise 4. Let $X = \mathbb{R} \times \mathbb{R}$, and define

$$\rho((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2, \\ 1 + |y_1 - y_2|, & \text{otherwise.} \end{cases}$$

Claim 4.1. *The function ρ is a metric.*

Proof. We need to check a number of properties:

- (1) $\rho((x, y), (x, y)) = |y - y| = 0$
- (2) $\rho(p, q) = \rho(q, p)$ because the definition of ρ is symmetric.
- (3) If $\rho((x_1, y_1), (x_2, y_2)) = 0$, then $x_1 = x_2$ (otherwise $\rho \geq 1$) and $|y_1 - y_2| = 0$. Hence, $(x_1, y_1) = (x_2, y_2)$.
- (4) $\rho((x_1, y_1), (x_2, y_2)) \geq |y_1 - y_2| \geq 0$
- (5) Triangle inequality: $\rho((x_1, y_1), (x_2, y_2)) \leq \rho((x_1, y_1), (x_3, y_3)) + \rho((x_3, y_3), (x_2, y_2))$

(a) If $x_1 = x_2$, then

$$\rho((x_1, y_1), (x_2, y_2)) = |y_1 - y_2| \leq |y_1 - y_3| + |y_3 - y_2| \leq \rho((x_1, y_1), (x_3, y_3)) + \rho((x_3, y_3), (x_2, y_2))$$

(b) If $x_1 \neq x_2$, then $x_1 \neq x_3$ or $x_3 \neq x_2$. Suppose without loss of generality that $x_1 \neq x_3$. Hence,

$$\rho((x_1, y_1), (x_2, y_2)) = 1 + |y_1 - y_2| \leq 1 + |y_1 - y_3| + |y_3 - y_2| \leq \rho((x_1, y_1), (x_3, y_3)) + \rho((x_3, y_3), (x_2, y_2))$$

□

Give X the topology induced by the metric ρ .

Lemma 4.2. *The subspace topology induced on $X \cap \{x\} \times \mathbb{R} = \mathbb{R}$ is equal to the usual topology on $\mathbb{R}$.*

A subset $U \subset X$ is open if and only if $U \cap \{x\} \times \mathbb{R} \subset \mathbb{R}$ is open in the usual topology, for every $x \in \mathbb{R}$.

A subset $C \subset X$ is compact if and only if the set $A_C = \{x \in \mathbb{R} : (x, y) \in C \text{ for some } y \in \mathbb{R}\}$ is finite, and $C \cap \{x\} \times \mathbb{R}$ is compact for every $x \in A_C$.

Proof. First, note that the restriction of ρ to $\{x\} \times \mathbb{R}$ gives the usual $\mathbb{R}$ -metric. Hence, the topology of $\{x\} \times \mathbb{R}$ is the usual topology.

Moreover, note that $\{x\} \times \mathbb{R}$ is an open set, because

$$\{x\} \times \mathbb{R} = \bigcup_{p \in \{x\} \times \mathbb{R}} B(p, 1/2)$$

Hence, if U is open, so is $U \cap (\{x\} \times \mathbb{R})$ in usual $\mathbb{R}$ topology. Conversely, if $U \cap (\{x\} \times \mathbb{R})$ is open for all $x \in \mathbb{R}$, then

$$U = \bigcup_{x \in \mathbb{R}} U \cap (\{x\} \times \mathbb{R})$$

is open as well.

Note that $\{x\} \times \mathbb{R}$ is also closed. That is because it is the complement of the open union

$$\bigcup_{y \neq x} \{y\} \times \mathbb{R}$$

Now, let $C \subset X$ be a compact set. Then $C \cap (\{x\} \times \mathbb{R})$ is a closed subset of C . As X is metric, it is Hausdorff. Hence, $C \cap (\{x\} \times \mathbb{R})$ is a compact subset of $\mathbb{R}$. Moreover, the open cover

$$C = \bigcup_{p \in C} B(p, 1/2)$$

is such that each open set $B(p, 1/2)$ is contained in a single strip $\{x\} \times \mathbb{R}$. Hence, any subcover has to contain at least one distinct open set for each element of A_C . Hence, as C is compact, A_C is finite.

Conversely, if A_C is finite, and $C \cap (\{x\} \times \mathbb{R})$ is compact for all $x \in A_C$, then C is a finite union of compact sets. Therefore, it is compact as well. □