

Proof. Fix $\epsilon > 0$. Then, for $\delta = \epsilon/n > 0$, if $d(x, y) < \delta$ then

$$|(f(p) + nd(x, p)) - (f(p) + nd(y, p))| = n|d(x, p) - d(y, p)| \leq nd(x, y) < \epsilon$$

for any $p \in X$. Hence,

$$g_n(x) = \inf_{q \in X} \{f(q) + nd(x, q)\} \leq f(p) + nd(x, p) < f(p) + nd(y, p) + \epsilon$$

for any $p \in X$. Therefore,

$$g_n(x) < \inf_{p \in X} \{f(p) + nd(y, p)\} + \epsilon = g_n(y) + \epsilon$$

Similarly,

$$g_n(y) < g_n(x) + \epsilon$$

Therefore, if $d(x, y) < \delta = \epsilon/n$, then $|g_n(x) - g_n(y)| < \epsilon$. Thus, g_n is a continuous function. $\square$

Claim 3.2. For any $x \in X$,

$$0 \leq g_1(x) \leq g_2(x) \leq \dots$$

and

$$f(x) = \sup_{n \in \mathbb{N}} g_n(x)$$

Proof. First, notice that $g_1(x) \geq 0$ because for any $q \in X$,

$$f(q) + d(x, q) \geq 0 + 0 = 0$$

Also notice that for any $x, p \in X$,

$$g_n(x) = \inf_{q \in X} \{f(q) + nd(x, q)\} \leq f(p) + nd(x, p) \leq f(p) + (n+1)d(x, p)$$

Hence,

$$g_n(x) \leq \inf_{p \in X} \{f(p) + (n+1)d(x, p)\} = g_{n+1}(x)$$

Moreover,

$$g_n(x) = \inf_{q \in X} \{f(q) + nd(x, q)\} \leq f(x) + nd(x, x) = f(x)$$

Thus, $\sup_{n \in \mathbb{N}} g_n(x) \leq f(x)$.

Finally, fix $x \in X$, and $\epsilon > 0$. We want to show that there exists a $n \in \mathbb{N}$ such that

$$f(q) + nd(q, x) + \epsilon > f(x), \text{ for any } q \in X$$

because this will imply that $g_n(x) + \epsilon > f(x)$ for any $\epsilon > 0$, which in turn implies that $\sup_{n \in \mathbb{N}} g_n(x) \geq f(x)$.

As f is lower semicontinuous, the set

$$U = \{y \in X | f(y) + \epsilon > f(x)\}$$

is an open neighborhood of x . Let $r = \inf_{y \notin U} d(y, x) > 0$. Pick n such that $nr > f(x)$.

Then,

(a) If $q \in U$, then

$$f(q) + nd(q, x) + \epsilon \geq f(q) + \epsilon > f(x)$$

(b) If $q \notin U$, then

$$f(q) + nd(q, x) + \epsilon > nd(q, x) \geq nr > f(x)$$

Hence, we are done! $\square$