

is the product of two real valued continuous maps, thus it is continuous itself.

Moreover, we know that $\rho_E(x) \geq 0$ for any $x \in X, E \subset X$. Hence,

$$0 \leq \rho_A(x) \leq \rho_A(x) + \rho_B(x)$$

what implies $f(x) \in [0, 1]$, what completes the proof of (1).

(2) If $a \in A$, then $\rho_A(a) = 0 \implies f(a) = 0$.

(3) If $b \in B$, then $\rho_B(b) = 0 \implies f(b) = \frac{\rho_A(b)}{\rho_A(b)} = 1$.

and we are done. $\square$

Proposition 1.4 (Constructive Urysohn's lemma). *If X is a metric space, then any two disjoint closed sets can be separated by a continuous function.*

Proof. By the claim above, given any two disjoint closed sets A, B , the function f defined by (1) separates the sets A and B . $\square$

25/25

Exercise 2. Assertions (a), (b), (d) are true, while assertion (c) is false.

(a) and (b): Both assertions are true. Indeed, if f_1, f_2 are real-valued functions, then,

$$\{f_1 + f_2 < \alpha\} = \cup_{\beta \in \mathbb{R}} \{f_1 < \beta\} \cap \{f_2 < \alpha - \beta\}$$

$$\{f_1 + f_2 > \alpha\} = \cup_{\beta \in \mathbb{R}} \{f_1 > \beta\} \cap \{f_2 > \alpha - \beta\}$$

Hence, as a finite intersection of open sets is open, and an arbitrary unions of open sets is open, we conclude that $f_1 + f_2$ is upper (lower) semicontinuous if both f_1 and f_2 are upper (lower) semicontinuous.

(c): This assertion is false. For example consider the functions

$$f_n(x) = \begin{cases} 1, & \text{if } \frac{1}{n+1} \leq x \leq 1/n \\ 0, & \text{else.} \end{cases}$$

Then, each function f_n is upper-semicontinuous, but $f(x) = \sum_n f_n(x)$ is such that

$$\{f < 1\} = (-\infty, 0]$$

is not open. Hence, f is not upper-semicontinuous.

(d): Let $g_n = f_1 + f_2 + \dots + f_n$. By (b), we know that finite sums of lower semicontinuous functions are lower-semicontinuous. In particular, each g_n is lower semicontinuous. Let $f(x) = \sum_n f_n(x)$. As f_i are non-negative,

$$f(x) > \alpha \iff \exists n \text{ such that } g_n(x) > \alpha$$

Therefore,

$$\{f > \alpha\} = \cup_{n \in \mathbb{N}} \{g_n > \alpha\}$$

is a union of open set, and therefore open. Hence, f is lower-semi-continuous.

25/25

Exercise 3. Let X be a metric space with metric d .

(1) Let $0 \leq g_1 \leq g_2 \leq \dots$ be a non-decreasing sequence of continuous functions. Let $f = \lim_{n \rightarrow \infty} g_n$. Then, for any $\alpha \in \mathbb{R}$,

$$\{f > \alpha\} = \cup_{n \in \mathbb{Z}} \{g_n > \alpha\}$$

is open. Hence, f is lower-semicontinuous.

(2) Conversely, suppose that $f : X \rightarrow [0, \infty]$ is lower-semicontinuous, and $f(p) < \infty$. Define

$$g_n(x) = \inf_{q \in X} \{f(q) + nd(x, q)\}$$

Claim 3.1. *The function g_n is continuous.*