

18.125, PROBLEM SET 6

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25/25 *Exercise 1* (Constructive Urysohn's Lemma). Let X be a metric space with metric ρ . For any non-empty subset $E \subset X$, define

$$\rho_E(x) = \inf\{\rho(x, y) \mid y \in E\}.$$

Claim 1.1. *The function ρ_E is uniformly continuous on X .*

Proof. Fix $\epsilon > 0$. Then, for $\delta = \epsilon > 0$, for any $x, y \in X$ such that $\rho(x, y) < \delta$,

$$\begin{aligned} |\rho_E(x) - \rho_E(y)| &= \left| \inf_{e \in E} \rho(x, e) - \inf_{e \in E} \rho(y, e) \right| \\ &\leq \left| \inf_{e \in E} [\rho(x, y) + \rho(y, e)] - \inf_{e \in E} \rho(y, e) \right| \\ &= |\rho(x, y) + \inf_{e \in E} \rho(y, e) - \inf_{e \in E} \rho(y, e)| \\ &= \rho(x, y) < \delta = \epsilon \end{aligned}$$

Hence, f is uniformly continuous. $\checkmark$

□

Claim 1.2. *Let $E \subset X$. Then*

$$\rho_E(e) = 0 \iff e \in \overline{E}$$

Proof. Note that

$$\begin{aligned} \rho_E(e) = 0 &\iff \forall \epsilon \in \mathbb{R}, \epsilon > 0, \exists x \in E \text{ s.t. } \rho(x, e) < \epsilon \\ &\iff \forall n \in \mathbb{N}, \exists x_n \in E \text{ s.t. } \rho(x_n, e) < \frac{1}{n} \\ &\iff \forall n \in \mathbb{N}, \exists x_i \in E \text{ s.t. the sequence } x_i \text{ converges to } e \\ &\iff e \in \overline{E} \end{aligned}$$

□

If $A, B \subset X$ are disjoint closed subsets, define

$$(1) \quad f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

Claim 1.3. *The function f defined above is such that*

- (1) *The function f is continuous, and $f(X) \subset [0, 1]$.*
- (2) *For any $a \in A$, $f(a) = 0$.*
- (3) *For any $b \in B$, $f(b) = 1$.*

That is, the function f separates A and B .

Proof. Note that $\rho_A(x) + \rho_B(x) \neq 0$ for all $x \in X$. Indeed, $\rho_A(x) + \rho_B(x) = 0 \implies \rho_A(x) = 0$ and $\rho_B(x) = 0 \implies x \in \overline{A} \cap \overline{B} = A \cap B = \emptyset$, contradiction!

Hence, the function $x \mapsto \rho_A(x) + \rho_B(x)$ is continuous and non-vanishing. Therefore, the map $x \mapsto (\rho_A(x) + \rho_B(x))^{-1}$ is also continuous. Finally, the map

$$x \mapsto f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$