

5.2 Using the "trick" we used in lecture to prove (p-measure) Fatou's Lemma, we find a subsequence f_{n_m} s.t.

$$\lim_{n \rightarrow \infty} \int f_{n_m} d\mu = \lim_{n \rightarrow \infty} \sup \int f_n d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Since $g_n \rightarrow g$ in $L^1(\mu)$, $g_n \rightarrow g$ in p-measure; thus by Thm 3.3.10 of Stroock, we can find a subsequence of g_{n_m} , say $g_{n_{m_i}}$, s.t. $g_{n_{m_i}} \rightarrow g$ p-a.e. Applying Fatou's Lemma to $h_n = g_n - f_n$, we have that

$$\begin{aligned} \int g d\mu - \lim_{n \rightarrow \infty} \int f_n d\mu &= \lim_{n \rightarrow \infty} \int g_{n_{m_i}} - f_{n_{m_i}} d\mu \\ &\geq \int \lim_{n \rightarrow \infty} g_{n_{m_i}} - f_{n_{m_i}} d\mu \\ &= \int g - \lim_{n \rightarrow \infty} f_{n_{m_i}} d\mu \\ &\geq \int g d\mu - \lim_{n \rightarrow \infty} \int f_n d\mu \end{aligned}$$

and thus $\lim_{n \rightarrow \infty} \int f_n d\mu \leq \int \lim_{n \rightarrow \infty} f_n d\mu$.

b. Suppose $f_n \rightarrow f$ p-a.e. Following the derivation of the DCT from Fatou's Lemma, we let $h_n = |f_n - f|$ so that $h_n \leq |f_n| + |f| \leq g_n + g$. Thus we have $h_n \leq g + \sup_{n \geq 1} g_n$, $\lim_{n \rightarrow \infty} f_n(x) = f$, $f \leq \sup_{n \geq 1} g_n$ so that by Fatou's Lemma,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \int f d\mu - \int f_n d\mu \right| &\leq \lim_{n \rightarrow \infty} \int |f_n - f| d\mu \leq \int \lim_{n \rightarrow \infty} |f_n - f| d\mu = 0 \\ \Rightarrow f_n &\rightarrow f \text{ in } L^1 \text{ and thus } \lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu. \end{aligned}$$

Given the preceding, the case of $f_n \rightarrow f$ in p-measure is handled in the same way as it was in the proof of Thm. 3.3.12.