

5.1 By Thm. 3.2.8, we have that $\mu(f = \infty) = 0$. Since $f \cdot 1_{f \geq \lambda} \rightarrow f \cdot 1_{f = \infty}$ and $\int_{f \geq \lambda} f d\mu = 0$, by the DCT,

20/20

$\int f \cdot 1_{f \geq \lambda} d\mu = \int_{f \geq \lambda} f d\mu \rightarrow 0$ as $\lambda \rightarrow \infty$. Notice that, by

Markov again, $\int_{f \geq \lambda} f d\mu \geq \lambda \mu(f \geq \lambda)$. Thus $\lambda \mu(f \geq \lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

As in Stroock, take $f(x) = \frac{1}{x(1+\log x)}$, $x \in (0, 1]$, $f(x) = \infty$ when $x = 0$.

Since $f'(x) = -\frac{1}{(1+\log x)^2} < 0$, f is decreasing on $[0, 1]$ to $f(1) = 1$.

For $\lambda \in [1, \infty)$, let $x_\lambda \in (0, 1]$ be s.t. $x_\lambda(1+\log x_\lambda) = \frac{1}{\lambda}$ so that

$\{x: f(x) \geq \lambda\} = [0, x_\lambda]$. Since f is decreasing, $\lim_{\lambda \rightarrow \infty} x_\lambda = 0$; moreover,

$x_\lambda \searrow 0$. Then this function satisfies condition 3.3.18

($\lambda \mu(f \geq \lambda) = \lambda x_\lambda = \frac{1}{1+\log x_\lambda} \rightarrow 0$ as $\lambda \rightarrow \infty$), but

$$\int_{(0,1]} f d\lambda_{\mathbb{R}} = \sum_{n=1}^{\infty} \int_{\frac{1}{n+1} \leq f < \frac{1}{n}} f d\lambda_{\mathbb{R}} \geq \sum_{n=1}^{\infty} n \mu(\frac{1}{n+1} \leq f < \frac{1}{n}) = \sum_{n=1}^{\infty} n(x_n - x_{n+1}) = \sum_{n=1}^{\infty} x_n = \infty$$

since $x_n \geq \frac{1}{n(1+\log n)}$ and $\sum_{n=2}^{\infty} \frac{1}{n(1+\log n)} = \infty$ (from pg. 62 of Rudin's

Principles of Mathematical Analysis).

Let $E_n = \{f \in (n-1, n]\}$. Then $\int f d\mu = \sum_{n=1}^{\infty} \int_{E_n} f d\mu \leq \sum_{n=1}^{\infty} n \mu(E_n)$

$$= \sum_{n=1}^{\infty} n \mu(f > n-1) - \sum_{n=1}^{\infty} \mu(f > n) = \sum_{n=0}^{\infty} \mu(f > n). \text{ Thus } \sum_{n=0}^{\infty} \mu(f > n) < \infty$$

implies that f is integrable. On the other hand,

$$\int_{f \geq N} f d\mu = \sum_{n=N}^{\infty} \int_{E_n} f d\mu \geq \sum_{n=N}^{\infty} (n-1) \mu(E_n) \geq \sum_{n=N}^{\infty} n \mu(E_n) - \mu(E) \geq \sum_{n=N}^{\infty} \mu(f > n) - \mu(E).$$

As $\mu(E)$ is finite, taking the limit of both sides as $N \rightarrow \infty$

gives us that the integrability of f implies $\sum_{n=0}^{\infty} \mu(f > n) < \infty$.

By Markov, we have that $\mu(f \geq \lambda) \leq \frac{1}{\lambda} \int f d\mu$. Thus $\mu(f \geq \lambda) \rightarrow 0$

as $\lambda \rightarrow \infty$. Let $\{f_n\}$ be s.t. $f_n(x) = 0$ if $f(x) \in [0, n]$ and

$f_n(x) = n$ if $f(x) > n$. Since $\lim_{n \rightarrow \infty} \mu(f_n > 0) = \lim_{n \rightarrow \infty} \mu(f > n) = 0$,

we see that $\{f_n\}$ are a decreasing sequence of simple functions bounded above by an integrable function f

s.t. $f_n \rightarrow 0$ in μ -measure. Thus by Thm. 3.3.12 of Stroock

(μ -measure DCT), we get that $\lim_{n \rightarrow \infty} \int f_n d\mu = \lim_{n \rightarrow \infty} n \mu(f > n) = 0$.

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