

4, 4 cont.

We also have $L_1(x) \leq L_2(x) \leq \dots \leq f(x) \leq \dots \leq U_2(x) \leq U_1(x)$

$\forall x \in [0, 1]$, from the construction of $\{L_k\}_1^\infty$, $L(x) = \lim_{k \rightarrow \infty} L_k(x)$

$U(x) = \lim_{k \rightarrow \infty} U_k(x)$, because monotonic and bounded

Sequences of functions converge pointwise.

It is clear that L and U are bounded ($L(x) \leq f(x) < \infty$

$\forall x \in [0, 1]$ and $U(x) \leq U_1(x) < \infty \forall x \in [0, 1]$) and that

L and U are measurable by monotone convergence

thm. (to show U is measurable we note that

$\{-U_k\}_1^\infty$ is an increasing seq. of measurable functions

so $\lim_{k \rightarrow \infty} -U_k = -U'(x)$ is measurable. It follows that

$-(\lim_{k \rightarrow \infty} -U_k) = \lim_{k \rightarrow \infty} U_k = -U'(x) = U(x)$ is measurable).

Therefore, we have $\lim_{k \rightarrow \infty} \int_{[0,1]} L_k(x) = \int_{[0,1]} L(x) = (R) \int_{[0,1]} f dx$ and

$\lim_{k \rightarrow \infty} \int_{[0,1]} U_k(x) = \int_{[0,1]} U(x) = (R) \int_{[0,1]} f dx$ by (1) and monotone

convergence thm.

Now note that since f is Riemann integrable,

(2) $(R) \int_{[0,1]} f dx = \int_{[0,1]} L(x) = \int_{[0,1]} U(x) = (R) \int_{[0,1]} f dx$. Because $L(x) \leq U(x) \forall x \in [0, 1]$,

we have (2) iff $L(x) = U(x)$ m.a.e.