

Note! unless otherwise stated all integrals are taken w.r.t. Lebesgue ^{measure} $\uparrow$

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Since f is bounded and Riemann integrable, there exists a sequence of finite, exact, non-overlapping rectangular covers $\{L_k\}_{k=1}^{\infty}$ of $[0, 1]$ such that if

$I \in L_{k+1}$, then $I \subset I'$ for some $I' \in L_k$ and such that

$\text{mesh}(L_k) < \frac{1}{k}$ for all $k \in \mathbb{N}$, and such that

$$(1) \quad \lim_{k \rightarrow \infty} L(L_k; f) = (R) \int_{[0,1]} f dx = \lim_{k \rightarrow \infty} U(L_k; f) = (R) \int_{[0,1]} f dx \quad (\forall k \in \mathbb{N})$$

Now define the simple functions U_k and L_k

such that $U_k(0) = L_k(0) = f(0)$ and for $x \neq 0$,

$$U_k(x) = \sup_{y \in I_x} f(y) \quad \text{where } I_x^* = I_x - \{a\} \text{ where } a \text{ is the left end point of } I_x \text{ and } x \in I_x^*$$

$$L_k(x) = \inf_{y \in I_x} f(y) \quad \text{where } I_x^* = I_x - \{a\} \text{ where } a \text{ is the left end point of } I_x \text{ and } x \in I_x^*$$

(Note that μ below is Lebesgue measure)

Then it is clear that $\int U_k d\mu = \sum_{d \in \text{Range}(U_k)} d \mu \{U_k = d\}$

$$\begin{aligned} \text{Let } \sup_{I \in L_k} &= \{I : \sup_{x \in I} f(x) = d\} \\ &= \sum_{d \in \text{Range}(U_k)} d \left(\sum_{I \in \sup_{I \in L_k}^d} \text{vol}(I) \right) \\ &= \sum_{\substack{\sup_{I \in L_k}^d \\ x \in \sup_{I \in L_k}^d}} U_k(x) \left(\sum_{I \in \sup_{I \in L_k}^d} \text{vol}(I) \right) \\ &= \sum_{\substack{\sup_{I \in L_k}^d \\ x \in \sup_{I \in L_k}^d}} \sup_{y \in I} f(y) \cdot \left(\sum_{I \in \sup_{I \in L_k}^d} \text{vol}(I) \right) \\ &= \sum_{I \in L_k} \sup_{x \in I} f(x) (\text{vol}(I)) = U(L_k; f) \end{aligned}$$

Similarly for $L(L_k; f)$, we have $L(L_k; f) = \int L_k d\mu$