

4.3 It suffices to show (1) $v(\phi) = 0$ and

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(2) v has the countable additivity property.

(1):

$$v(\phi) = \int_{\phi} f \, d\mu = \int_E f \cdot 1_{\phi} \, d\mu = \int_E 0 \, d\mu = 0$$

(2): Let $\{\Gamma_n\}_1^{\infty}$ be a countable, disjoint collection of sets in $\mathcal{B}$. It is clear that:

$$f \cdot 1_{\{\cup_1^{\infty} \Gamma_n\}} = \sum_{n=1}^{\infty} f \cdot 1_{\Gamma_n}$$

By definition we have:

$$v(\cup_1^{\infty} \Gamma_n) = \int_E f \cdot 1_{\{\cup_1^{\infty} \Gamma_n\}} \, d\mu, \quad v(\Gamma_n) = \int_E f \cdot 1_{\Gamma_n} \, d\mu$$

To show (2) we prove a more general

lemma: If $f_n: X \rightarrow [0, \infty]$ is measurable, for

$n = 1, 2, 3, \dots$, and $f(x) = \sum_{n=1}^{\infty} f_n(x)$, for $x \in \Gamma \in \mathcal{B}$, then

$$\int_{\Gamma} f \, d\mu = \sum_{n=1}^{\infty} \int_{\Gamma} f_n \, d\mu.$$

Pf: By the linearity lemma (Lemma 3.2.7

Stroock p. 46) we have that $\int_{\Gamma} (f_1 + f_2) \, d\mu = \int_{\Gamma} f_1 \, d\mu + \int_{\Gamma} f_2 \, d\mu$

Now consider $\phi_M = f_1 + \dots + f_M$. The sequence