

4.2 Suppose $\phi \in L^1(\mu)$ and $\mu(\{\phi < 0\}) = 0$. Then

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it is clear that $\int \phi d\mu \geq 0$ for each $\Gamma \in \mathcal{B}$, because

we may find a sequence $\{\psi_n\}_{n=1}^{\infty}$ of non-negative simple measurable functions such that $\psi_n \uparrow \phi$ pointwise as $n \rightarrow \infty$

so by monotone convergence Thm, we have:

$\int \phi d\mu = \lim_{n \rightarrow \infty} \int \psi_n d\mu \geq 0$. Conversely suppose that for each $\Gamma \in \mathcal{B}$

$\int \phi d\mu = 0$. Then take $\Gamma_n = \{\phi \leq -\frac{1}{n}\}$ and note that

$$0 \leq \int_{\Gamma_n} \phi d\mu \leq -\frac{1}{n} \mu(\Gamma_n)$$

and thus $0 = \mu(\Gamma_n) \uparrow \mu(\{\phi < 0\})$.

Now the preceding arguments reduce the problem at hand if we set $\phi = g - f$. In this case,

suppose that $\mu(\{\phi < 0\}) = \mu(\{g - f < 0\}) = 0$ iff

$\int_{\Gamma} (g - f) d\mu \geq 0$ for each $\Gamma \in \mathcal{B}$. Then $-\int_{\Gamma} (g - f) d\mu \leq 0$ for each

$\Gamma \in \mathcal{B}$ iff $\mu(\{\phi > 0\}) = 0 = \mu(\{-(g - f) > 0\})$. Thus $\int_{\Gamma} f d\mu = \int_{\Gamma} g d\mu$,

or equivalently, $\int_{\Gamma} f - g d\mu = 0 = -\int_{\Gamma} f - g d\mu = \int_{\Gamma} g - f d\mu$ iff

$\mu(\{g - f < 0\}) = 0 = \mu(\{f - g < 0\}) = \mu(\{g - f \neq 0\})$ which by

definition means that $f \sim g$. Q.E.D.