

4.1 cont.

4.1 similar. Thus the converse follows.

Now suppose that $\{f \geq a\} \in \mathcal{B}$ for each $a \in \mathbb{D}$. Given $b \in \mathbb{R}$, choose a sequence $\{a_n\}_1^\infty \subseteq \mathbb{D} \cap (b, \infty)$ such that $a_n \downarrow b$. It is clear $\{f > b\} = \bigcup_1^\infty \{f \geq a_n\} \in \mathcal{B}$, hence the current case reduces to the previous.

We may consider the cases " $<$ " and " $\leq$ " by considering $-f$ in place of f .

Suppose that g is a second $\mathbb{R}$ -valued function on $(E, \mathcal{B})$. First we show $\{f < g\} \in \mathcal{B}$.

Let $\mathbb{Q}$ denote the set of rationals in $\mathbb{R}$, then we have:

$$\{f < g\} = \bigcup_{p \in \mathbb{Q}} (\{f < p\} \cap \{g > p\}) \in \mathcal{B}$$

The correctness of this assertion is clear. If

$x \in \{f < g\}$, then $x \in \{f < p\} \cap \{g > p\}$ where $f(x) < p < g(x)$,

the existence of p follows from the Archimedean principle

and the density of $\mathbb{Q}$. Conversely, if $x \in \{f < p\} \cap \{g > p\}$ we have

$f(x) < p < g(x)$, so $x \in \{f < g\}$. It is clear that $\bigcup_{p \in \mathbb{Q}} (\{f < p\} \cap \{g > p\}) \in \mathcal{B}$

from the preceding arguments. $\{g > f\} \in \mathcal{B}$ by symmetry.