

4.5 Let $E_n = \{x \in E : |f(x)| \leq n\}$. It is clear that

20/20 $\bigcup E_n = E$. Then since $\mu(f = \infty) = 0$, (by Markov's Inequality Thm 3.2.8) it follows that $g_n = f \cdot 1_{E_n} \uparrow f$ pointwise μ a.e.. Additionally it is clear that $|g_n| \leq f$ for $n \geq 1$, so by Lebesgue's Dominated Convergence Thm

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$\lim_{n \rightarrow \infty} \int g_n d\mu = \int f$, as it is clear $\uparrow$ p.t. wise μ a.e.

that $g_n \uparrow f$ and that $\{g_n\}$ is a non-negative sequence of measurable functions. By definition

of the limit there exists n_0 such that $n > n_0$ implies

that $\int_{E \setminus E_n} f d\mu = \int f d\mu - \int_{E_n} f d\mu < \frac{\epsilon}{2}$. Choose $\delta = \frac{\epsilon}{2(n_0+1)}$

Then we have for $\mu(\Gamma) < \delta = \frac{\epsilon}{2(n_0+1)}$:

$$\int_{\Gamma} f d\mu = \int_{\Gamma \cap E_{n_0+1}} f d\mu + \int_{\Gamma \cap (E \setminus E_{n_0+1})} f d\mu < \frac{(n_0+1)\epsilon}{2(n_0+1)} + \frac{\epsilon}{2} = \epsilon$$

We have demonstrated the desired result.