

18.125 PSET 4

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The forward direction follows from the

definition of f as a measurable function. Conversely,

suppose that for each and every $a \in D \subset \mathbb{R}$, with D

dense, $\{f > a\} \in \mathcal{B}$. It is clear that $\{f > a\} \in \mathcal{B} \Leftrightarrow f^{-1}(\Gamma) \in \mathcal{B}$ for every

$\Gamma \in \mathcal{C} \equiv \{(a, \infty] : a \in D\}$. By the homework exercise

3.1.10 it suffices to show that $\mathcal{C}$ generates the

Borel sets over $\mathbb{R}$ to show that f is measurable.

Since the Borel sets may be expressed as

countable unions/intersections the intervals

(c, d) such that $-\infty < c < d < \infty$, we should show

that $(c, d) \in \sigma(\mathbb{R}; \mathcal{C})$ for every $-\infty < c < d < \infty$.

Given $-\infty < c < d < \infty$, we may find $\{a_n\} \subseteq (c, \infty) \cap D$

and $\{b_n\} \subseteq (-\infty, d) \cap D$ so that $a_n \downarrow c$ and $b_n \uparrow d$, from

the density of D , which implies:

$$(c, d) = \left(\bigcup_1^\infty (a_n, \infty] \right) \cap \left(\bigcap_1^\infty (b_n, \infty] \right)^c \in \mathcal{B}$$

Correctness is evident because $\bigcup_1^\infty (a_n, \infty] = (a, \infty]$ and

$\left(\bigcap_1^\infty (b_n, \infty] \right)^c = \bigcap_1^\infty [\infty, b_n] = [\infty, d)$. The case for $c = -\infty$ or $d = \infty$ is