

that $F \subseteq \Gamma \subseteq G$ and $\mu(G \setminus F) < \epsilon$. Then, G^C is closed, F^C is open, $G^C \subseteq \Gamma^C \subseteq F^C$, and $\mu(F^C \setminus G^C) = \mu(F^C \cap G) = \mu(G \setminus F) < \epsilon$. So $\Gamma^C \in \mathcal{B}$. Finally, choose $\{\Gamma_n\}_{n=1}^\infty \subseteq \mathcal{B}$ and $\epsilon > 0$, and let $\Gamma = \bigcup_{n=1}^\infty \Gamma_n$. Then, for each Γ_n , we have closed F_n and open G_n such that $F_n \subseteq \Gamma_n \subseteq G_n$, and $\mu(G_n \setminus F_n) < 2^{-n}\epsilon$. Let $G = \bigcup_{n=1}^\infty G_n$, $F'_n = \bigcup_{m=1}^n F_m$, and $F = \bigcup_{n=1}^\infty F_n$. Clearly, G is open, and F'_n is closed for all n . Also, $F \subseteq \Gamma \subseteq G$, and by Theorem 3.1.6, $\mu(G \setminus F) < \epsilon$. Since $F'_n \nearrow F$, again by Theorem 3.1.6, $\mu(F'_n) \nearrow \mu(F)$, and there exists n_0 such that $\mu(F \setminus F'_n) < \epsilon$ for all $n \geq n_0$. Combining these, $\mu(G \setminus F'_n) < 2\epsilon$ for large enough n , and $\Gamma \in \mathcal{B}$. Thus, $\mathcal{B}$ is a σ -algebra.

Furthermore, choose Γ to be open. Then, choose $G = \Gamma$, and let

$$F_n = \{x \in E : \inf_{y \in \Gamma^C} \rho(x, y) \geq 2^{-n}\}.$$

Clearly, F_n^C is open, so F_n is closed. Also, $F_n \subseteq F_{n+1}$, and $F_n \nearrow \Gamma$. Choose $\epsilon > 0$. Using Theorem 3.1.6, $\mu(G \setminus F_n) < \epsilon$ for large enough n , and $\mu(G \setminus F_n) < \epsilon$, implying $\Gamma \in \mathcal{B}$.

Now, choose $\Gamma \in \overline{\mathcal{B}}_E^\mu$ and $\epsilon > 0$. Then, there exist $A, B \in \mathcal{B}_E$ such that $A \subseteq \Gamma \subseteq B$ and $\mu(B \setminus A) = 0$. Since $A, B \in \mathcal{B}$, there exist F_A, F_B, G_A, G_B such that $\mu(G_A \setminus F_A) < \epsilon$ and $\mu(G_B \setminus F_B) < \epsilon$. Using Theorem 3.1.6, $\mu(A \setminus F_A) < \epsilon$, $\mu(G_B \setminus B) < \epsilon$, and $\mu(G_B \setminus F_A) = \mu((G_B \setminus B) \cup (B \setminus A) \cup (A \setminus F_A)) < 2\epsilon$. Thus, $F = F_A$ and $G = G_B$ satisfy $F \subseteq \Gamma \subseteq G$ and $\mu(G \setminus F) < 2\epsilon$, $\Gamma \in \mathcal{B}$, and $\mathcal{B} = \overline{\mathcal{B}}_E^\mu$. $\square$

Problem 3.5

Proof. Let $B = \{\Gamma \in \mathcal{B}_2 : \Phi^{-1}(\Gamma) \in \mathcal{B}_1\}$; I will prove that B is a σ -algebra. It is obvious that $\emptyset \in B$. Also, since $\Phi^{-1}(\Gamma^C) = \Phi^{-1}(\Gamma)^C \in \mathcal{B}_1$, and B is closed under complementation. Also, consider $\{\Gamma_n\}_{n=1}^\infty$ and $\Gamma = \bigcup_{n=1}^\infty \Gamma_n$; then $\Phi^{-1}(\Gamma) = \bigcup_{n=1}^\infty \Phi^{-1}(\Gamma_n) \in \mathcal{B}_1$, and B is closed under countable unions. Thus, B is a σ -algebra. Since $B \supseteq \mathcal{C}$, and $\mathcal{C}$ generates $\mathcal{B}_2$, $\mathcal{B}_2 \subseteq B$. Thus, for every $\Gamma \in \mathcal{B}_2$, $\Gamma \in B$, and $\Phi^{-1}(\Gamma) \in \mathcal{B}_1$.

Let $\mathcal{C}$ be the topology (collection of open sets) on topological space E_2 . Since Φ is continuous, the inverse image Φ^{-1} maps open sets to open sets, and $\Phi^{-1}(G) \in \mathcal{B}_1$ for all $G \in \mathcal{C}$. Since the Borel algebra $\mathcal{B}_2$ is generated from $\mathcal{C}$, we can apply the first part to show that $\Phi^{-1}(G) \in \mathcal{B}_1$ for all $G \in \mathcal{B}_2$.

Finally, choose $x \in \mathbb{R}^N$, and let $\Phi : \Gamma \in \mathbb{R}^N \rightarrow \Gamma - x$. Clearly, Φ is continuous, and $\Phi^{-1}(\Gamma) = x + \Gamma$. Then, applying the second part, $x + \Gamma \in \mathcal{B}_{\mathbb{R}^N}$ for all $x \in \mathbb{R}^N$ and $\Gamma \in \mathcal{B}_{\mathbb{R}^N}$. $\square$