

Clearly, $\mathcal{P}(B_n)$ is a finite collection of sets in $\mathcal{B}$, so there exist an infinite number of sets V in $\{U_n\}_{n=3}^\infty$ such that $V \neq \cup \Gamma$ and $V^C \neq \cup \Gamma$, for all $\Gamma \in \mathcal{P}(B_n)$; choose such a set V (V and V^C are both non-empty). Several possibilities exist:

1. $V \cap (\cup B_n)^C = \emptyset$. For all $U'_m \in B_n$ such that $U'_m \subset V$, replace V with $V \setminus U'_m$. The resulting V cannot be empty, since otherwise $V = \cup \Gamma$ for some $\Gamma \in \mathcal{P}(B_n)$. Let $B_{n+1} = \{U'_1 \setminus V, \dots, U'_n \setminus V, V\}$. Clearly, all sets in B_{n+1} are disjoint, and since $U'_m \not\subseteq V$ for all $1 \leq m \leq n$, all the sets in B_{n+1} are non-empty. Also, $\cup B_{n+1} = \cup B_n \subset E$.
2. $V \subset (\cup B_n)^C$. Then, let $B_{n+1} = \{U'_1, \dots, U'_n, V\}$. B_{n+1} clearly is still a collection of non-empty disjoint sets, and since $V \neq (\cup B_n)^C$, $\cup B_{n+1}$ is a proper subset of E .
3. $V \supset (\cup B_n)^C$. Then $V^C \cap (\cup B_n)^C = \emptyset$, and use the first case with V^C (again, if $U'_m \subset V^C$ for some $U'_m \in B_n$, need to remove U'_m from V^C).
4. $V \cap (\cup B_n)^C \neq \emptyset$, but $V \not\subseteq (\cup B_n)^C$ and $V \not\supset (\cup B_n)^C$. Then $V \cap (\cup B_n)^C \subset (\cup B_n)^C$, and use the second case.

Thus, in any case, we can construct a larger collection B_{n+1} of disjoint sets that satisfies $\cup B_{n+1} \subset E$. Using induction, we can construct a countably infinite collection $B' = \{U'_n\}_{n=1}^\infty \subset \mathcal{B}$ of disjoint sets.

Consider the power set $\mathcal{P}(B')$; by Cantor's theorem, the power set of an infinite set is uncountable. For each $\Gamma \in \mathcal{P}(B')$, Γ is an at most countable collection of sets in B' . Thus, $\cup \Gamma$ is a countable union of sets in B' , so $\cup \Gamma$ must be in $\mathcal{B}$. Furthermore, for all $\Gamma_1, \Gamma_2 \in \mathcal{P}(B')$ such that $\Gamma_1 \neq \Gamma_2$, I claim that $\cup \Gamma_1 \neq \cup \Gamma_2$.

To prove this claim, since $\Gamma_1 \neq \Gamma_2$, there exists $U'_n \in B'$ such that $U'_n \in \Gamma_1 \cap \Gamma_2^C$ or $U'_n \in \Gamma_1^C \cap \Gamma_2$. Since $U'_n \cap U'_m = \emptyset$ for all $m \neq n$, $U'_n \cap (\cup \Gamma_2) = \emptyset$ for the first case, or $U'_n \cap (\cup \Gamma_1) = \emptyset$ for the second case, so $\cup \Gamma_1 \neq \cup \Gamma_2$. This proves the claim.

Since $\mathcal{B}$ contains uncountably many sets, $\mathcal{B}$ cannot be countably infinite. □

Problem 3.4

Proof. Define $\mathcal{B} = \{\Gamma \subseteq E \text{ s.t. } \forall \epsilon > 0, \exists \text{ closed } F, \text{ open } G : F \subseteq \Gamma \subseteq G, \mu(G \setminus F) < \epsilon\}$. Choose $\Gamma \in \mathcal{B}$ and $\epsilon > 0$; by construction, we have F, G such that $F \subseteq \Gamma \subseteq G$ and $\mu(G \setminus F) < \epsilon$. Since the Borel field $\mathcal{B}_E$ contains all open and closed sets, $F, G \in \mathcal{B}_E$. Since we can choose F and G to make $\mu(G \setminus F)$ arbitrarily small, $\Gamma \in \overline{\mathcal{B}}_E^\mu$, and $\mathcal{B} \subseteq \overline{\mathcal{B}}_E^\mu$.

For $\mathcal{B} = \overline{\mathcal{B}}_E^\mu$, $\mathcal{B}$ must be a σ -algebra. So, let us check that $\mathcal{B}$ is a σ -algebra. Obviously, $\emptyset \in \mathcal{B}$, since we can choose $F = \emptyset$ and $G = \emptyset$ to satisfy $F \subseteq \emptyset \subseteq G$ and $\mu(G \setminus F) = 0 < \epsilon$ for all $\epsilon > 0$. Also, if $\Gamma \in \mathcal{B}$, for any $\epsilon > 0$, we can choose closed F and open G such