

additivity is trivially satisfied. Suppose infinitely many Γ_n contain at least one element. Then, since these sets are disjoint, the union contains an infinite number of elements, and $\mu_1(\cup_{n=1}^{\infty} \Gamma_n) = \infty$, and $\sum_{n=1}^{\infty} \mu_1(\Gamma_n) \leq \sum_{n=1}^{\infty} 1 = \infty$. Finally, consider the case where only a finite number of sets contain at least one element. Then, the total number of non-empty sets in $\{\Gamma_n\}_{n=1}^{\infty}$ is finite, and $\mu_1(\cup_{n=1}^{\infty} \Gamma_n) = \mu_1(\cup_{n:|\Gamma_n|>0} \Gamma_n) = \sum_{n:|\Gamma_n|>0} \mu_1(\Gamma_n) = \sum_{n=1}^{\infty} \mu_1(\Gamma_n)$. Thus, μ_1 satisfies countable additivity for all cases and is a measure.

Now, I need to prove μ_2 is a measure. Obviously, $\mu_2(\emptyset) = 0$ by construction. Now, to show countable additivity, choose a countable collection of disjoint sets $\{\Gamma_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$. First, suppose $0 \notin \Gamma_n$ for all Γ_n in the collection; then, $\mu_2(\cup_{n=1}^{\infty} \Gamma_n) = 0 = \sum_{n=1}^{\infty} \mu_2(\Gamma_n)$, satisfying countable additivity trivially. Otherwise, there exists $m \in \mathbb{Z}^+$ such that $0 \in \Gamma_m$. Since $\Gamma_m \cap \Gamma_n = \emptyset$ for all $m \neq n$, $0 \notin \Gamma_n$ for all $n \neq m$, and $\sum_{n=1}^{\infty} \mu_2(\Gamma_n) = \mu_2(\Gamma_m) = 1$, and $\mu_2(\cup_{n=1}^{\infty} \Gamma_n) = 1$, since the union obviously contains 0. Thus, countable additivity is satisfied, and μ_2 is a measure.

10/10 (b) Clearly $\emptyset$ is countable, so $\emptyset \in \mathcal{B}$. Suppose $U \in \mathcal{B}$; by definition, either U is countable, or U^C is countable. Then, $U^C \in \mathcal{B}$, also by construction, since U^C also satisfies either being countable or having a countable complement. Now, let $\{U_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$. Split this collection into $\{A_n\}_{n=1}^{\infty}$, where A_n is at most countable, and $\{B_n\}_{n=1}^{\infty}$, where B_n^C is at most countable. Then, $\cup_{n=1}^{\infty} U_n = (\cup_{n=1}^{\infty} A_n) \cup (\cup_{n=1}^{\infty} B_n)$. Clearly, $\cup_{n=1}^{\infty} A_n$ is countable, being a countable union of countable sets. Also, $\cup_{n=1}^{\infty} B_n = (\cap_{n=1}^{\infty} B_n^C)^C$; since B_n^C is countable, $\cap_{n=1}^{\infty} B_n^C$ is countable, and $\cup_{n=1}^{\infty} B_n$ has a complement that is at most countable. If $\{A_n\}$ is empty, or $\{B_n\}$ is empty, $\{U_n\}$ clearly is in $\mathcal{B}$. Otherwise, since $(\cup_{n=1}^{\infty} A_n) \cup (\cup_{n=1}^{\infty} B_n)^C \subseteq (\cup_{n=1}^{\infty} B_n)^C$, the complement of $\cup_{n=1}^{\infty} U_n = \cup_{n=1}^{\infty} A_n \cup (\cup_{n=1}^{\infty} B_n)$ is also at most countable, and $\cup_{n=1}^{\infty} U_n \in \mathcal{B}$. Thus, $\mathcal{B}$ is a σ -algebra.

Clearly, $\mu(\emptyset) = 0$, since $\emptyset$ is at most countable. Now, suppose $\{U_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$; divide this sequence into a sequence of those sets $\{A_n\}_{n=1}^{\infty}$ that are countable and those sets $\{B_n\}_{n=1}^{\infty}$ that have countable complements. If $\{B_n\}$ is empty, $\mu(\cup_{n=1}^{\infty} U_n) = \mu(\cup_{n=1}^{\infty} A_n) = 0 = \sum_{n=1}^{\infty} \mu(U_n)$, satisfying countable additivity. Otherwise, $\mu(\cup_{n=1}^{\infty} U_n) = 1$, as $\cup_{n=1}^{\infty} U_n$ has a countable complement, using the reasoning from the proof that $\mathcal{B}$ is a σ -algebra. In the situation that $\{B_n\}$ is not empty, I claim that multiple sets B_m and B_n cannot both be disjoint and have countable complements. Since for $m \neq n$, $B_m \cap B_n = \emptyset$, $B_m^C \cup B_n^C = \mathbb{R}$. Since B_m^C and B_n^C are countable, $\mathbb{R}$ must be countable, which is a contradiction. Thus, $\sum_{n=1}^{\infty} \mu(U_n) = 1 + \sum_{n=1}^{\infty} \mu(A_n) = 1$, which confirms countable additivity. Thus, μ is a measure. $\square$

10/10 Problem 3.3

Proof. Suppose $\mathcal{B} = \{U_n\}_{n=1}^{\infty}$ were a countably infinite σ -algebra over E . For the purposes of this problem, assume $U_1 = \emptyset$, and $U_2 = E$; then U_n and U_n^C are non-empty for all $n \geq 3$. Construct a new sequence $B' = \{U'_n\}_{n=1}^{\infty}$ of non-empty disjoint sets as follows. Start with $B_1 = \{U'_1 = U_3\}$; clearly, B_1 is a collection of non-empty, disjoint sets, and $\cup B_1 = U_3 \subset E$. Suppose that $B_n = \{U'_1, \dots, U'_n\}$ is a collection of non-empty disjoint sets such that $\cup B_n \subset E$.