

$B_n \subset G_n$, with closure $\overline{B_n}$ also a proper subset of G_n (just choose a ball a little smaller than the largest possible open ball for that point). Since B_n is a non-empty open subset of $\mathbb{R}$, $G_{n+1} = B_n \cap U_n$ is also non-empty (and open). Furthermore, $G_{n+1} \subseteq B_n$, and $\overline{B_{n+1}} \subset \overline{B_n}$. Since $\mathbb{R}$ is complete, $\cap_{n=1}^{\infty} \overline{B_n}$ is not empty. Since $\overline{B_n} \subset (G \cap U_n)$, $\cap_{n=1}^{\infty} \overline{B_n} \subseteq \cap_{n=1}^{\infty} (G \cap U_n) = G \cap (\cap_{n=1}^{\infty} U_n)$, and $G \cap (\cap_{n=1}^{\infty} U_n)$ is not empty. Since this intersection is non-empty for every non-empty open subset G of $\mathbb{R}$, $\cap_{n=1}^{\infty} U_n$ is also dense in $\mathbb{R}$.

Step 2: Assume $\mathbb{Q}$ is a G_δ set. Thus, there exist open sets $\{U_n\}_{n=1}^{\infty}$ such that $\mathbb{Q} = \cap_{n=1}^{\infty} U_n$. Since $\mathbb{Q}$ is dense, for any open subset G of $\mathbb{R}$, $G \cap \mathbb{Q} \neq \emptyset$, so $U_n \cap G \neq \emptyset$ for all n , and all U_n are dense as well. Define $Q_{1/n}$ as in part (b); $\mathbb{R} \setminus Q_{1/n}$ is open (see proof in part (b)).

I claim $\mathbb{R} \setminus Q_{1/n}$ is dense: let G be a non-empty open subset of $\mathbb{R}$. Suppose $G \subseteq \mathbb{R} \setminus Q_{1/n}$, in which case the intersection is trivially non-empty. Otherwise, there exists $m \in \mathbb{Z}$ such that $m/n \in G$. Then, since G is open, there exists a non-empty open ball with radius $r < 1/n$ in G , containing m/n . Thus, $m/n + r/2$ is in G and is not in $Q_{1/n}$. Thus, $G \cap \mathbb{R} \setminus Q_{1/n}$ is not empty, and $\mathbb{R} \setminus Q_{1/n}$ is dense.

Since $\mathbb{R} \setminus Q_{1/n}$ is dense and open, and $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = (\cap_{n=1}^{\infty} U_n) \cap (\cap_{n=1}^{\infty} (\mathbb{R} \setminus Q_{1/n}))$, using Step 1, $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})$ is also dense. However, $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})$ is obviously empty, which is a contradiction (an empty set cannot be dense, since the intersection of any open set and the empty set is empty). Thus, $\mathbb{Q}$ is not a G_δ set.

Step 3: Choose $n \geq 1$, and let $B_n = \cup_{m=-n^2}^{n^2} \cap_{\ell=1}^{\infty} B(m/n, 1/\ell)$, where $B(m/n, 1/\ell)$ is the open ball centered at m/n with radius $1/\ell$. Since $B_n = \cap_{\ell=1}^{\infty} \cup_{m=-n^2}^{n^2} B(m/n, 1/\ell)$ is the infinite intersection of open sets, $B_n \in \mathfrak{G}_\delta$ for all n . Also, let $q = a/b \in \mathbb{Q}$ ($b \in \mathbb{Z}^+$, $a \in \mathbb{Z}$). Choose positive integer $c \geq |a|/b^2$; then, $|a| \leq b^2 c$, $|ac| \leq (bc)^2$, and $q \in B_{bc}$. Thus, $\mathbb{Q} \subseteq \cup_{n=1}^{\infty} B_n$. To prove the other direction, let $x \in \cup_{n=1}^{\infty} B_n$. Then, there exists n such that $x \in B_n$. Using the definition of B_n , there exists integer m such that $|m| \leq n^2$, and $x \in \cap_{\ell=1}^{\infty} B(m/n, 1/\ell)$. Suppose $x \neq m/n$; then $|x - m/n| > 0$, and $x \notin B(m/n, 1/\ell)$ for $\ell > 1/|x - m/n|$. Since this results in a contradiction, x must equal m/n , in which case $x \in \mathbb{Q}$. Thus, $\cup_{n=1}^{\infty} B_n \subseteq \mathbb{Q}$. So $\mathbb{Q} = \cup_{n=1}^{\infty} B_n$, and using Step 2, $\mathfrak{G}_\delta$ is not closed under countable unions. $\square$

Problem 3.2

Proof. (a) For $\mathcal{B}$ to be a σ -algebra, $\emptyset \in \mathcal{B}$, and $\mathcal{B}$ must be closed under complementation and countable unions. Since $\emptyset \subset \mathbb{R}^N$, $\emptyset \in \mathcal{B}$ by definition. Suppose $U \subseteq \mathbb{R}^N$. Since $U^C \subseteq \mathbb{R}^N$ by definition (complement taken with respect to $\mathbb{R}^N$), $U^C \in \mathcal{B}$. Finally, suppose $\{U_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$. Then, $\cup_{n=1}^{\infty} U_n \subseteq \mathbb{R}^N$, since $U_n \subseteq \mathbb{R}^N$, and $\cup_{n=1}^{\infty} U_n \in \mathcal{B}$. Thus, $\mathcal{B}$ is a σ -algebra.

Now, for μ_1 to be a measure on $(\mathbb{R}^N, \mathcal{B})$, $\mu_1(\emptyset) = 0$, and $\mu_1(\cup_{n=1}^{\infty} \Gamma_n) = \sum_{n=1}^{\infty} \mu_1(\Gamma_n)$ for all $\{\Gamma_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$ such that $\Gamma_m \cap \Gamma_n = \emptyset$ for $m \neq n$. Clearly $\emptyset$ is finite and has zero elements in it, so $\mu_1(\emptyset) = 0$ by construction. Now, suppose $\{\Gamma_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$ such that $\Gamma_m \cap \Gamma_n = \emptyset$ for $m \neq n$. Suppose all Γ_n are finite; otherwise, $\cup_{n=1}^{\infty} \Gamma_n$ is also infinite, and countable