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18.125 Homework #3

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Problem 3.1

8/8 *Proof.* (a) To be a topology, $\emptyset, E \in \tau$, and τ must be closed over (finite and infinite) unions and finite intersections. Clearly, $\emptyset$ and $E = \mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n)$ are in τ . Suppose $u_1 = \bigcup_{\alpha} [a_{\alpha}, b_{\alpha})$, $u_2 = \bigcup_{\beta} [a_{\beta}, b_{\beta}) \in \tau$. Then, $u_1 \cap u_2 = \bigcup_{\alpha, \beta} ([a_{\alpha}, b_{\alpha}) \cap [a_{\beta}, b_{\beta}))$. The intersection of two half-open intervals is another half-open interval:

$$[a, b) \cap [c, d) = \begin{cases} [\max\{a, c\}, \min\{b, d\}), & \max\{a, c\} < \min\{b, d\}; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Thus, $u_1 \cap u_2$ is a union of sets in τ_0 , so $u_1 \cap u_2 \in \tau$, and by induction, τ is closed over finite intersections. In addition, for any $\{u_{\alpha}\} \subseteq \tau$, any union $\bigcup_{\alpha} u_{\alpha}$ is a union of unions of sets in τ_0 , so $\bigcup_{\alpha} u_{\alpha} \in \tau$ as well. Thus, τ is a topology.

7/7 (b) Given $n \in \mathbb{Z}^+$, let $Q_{1/n} = \{m/n : m \in \mathbb{Z}\}$. Then, $\mathbb{Q} = \bigcup_{n=1}^{\infty} Q_{1/n}$, and $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{n=1}^{\infty} (\mathbb{R} \setminus Q_{1/n})$. So, I will prove $\mathbb{R} \setminus Q_{1/n}$ is open and use the definition of $\mathfrak{G}_{\delta}$ to show $\mathbb{R} \setminus \mathbb{Q}$ is a G_{δ} set.

Choose $n \in \mathbb{Z}^+$, and let $x \in \mathbb{R} \setminus Q_{1/n}$. Then, by the least upper bound property, there exists $m \in \mathbb{Z}$ such that $\frac{m}{n} < x < \frac{m+1}{n}$. Thus, the open ball $B(x, \min\{x - \frac{m}{n}, \frac{m+1}{n} - x\})$ contains x and is contained in $\mathbb{R} \setminus Q_{1/n}$. Thus, x is an interior point of $\mathbb{R} \setminus Q_{1/n}$. Since this is true for all $x \in \mathbb{R} \setminus Q_{1/n}$, $\mathbb{R} \setminus Q_{1/n}$ is open.

Since $\mathbb{R} \setminus Q_{1/n}$ is open, and $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{n=1}^{\infty} (\mathbb{R} \setminus Q_{1/n})$, using the definition of $\mathfrak{G}_{\delta}$, $\mathbb{R} \setminus \mathbb{Q}$ is a G_{δ} set.

15/15 (c) Step 1: Choose a non-empty open subset $G \subset \mathbb{R}$. Then, to prove $\bigcap_{n=1}^{\infty} U_n$ to be dense, I need to show that the intersection with G is not empty. Since U_1 is dense, $G_1 = G \cap U_1$ is not empty; since U_1 is open, and G is open, G_1 is also open. Suppose for $n \geq 1$, G_n is non-empty and open. Then, around any point in G_n , there exists a non-empty open ball