

18.125 - PROBLEM SET 2

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Problem 1. Let $A = \{a_1, a_2, \dots\}$ be a countable set. We can pick rectangles $R_i(\epsilon)$ such that $\text{vol}(R_i(\epsilon)) < \epsilon/2^i$ and $a_i \in R_i(\epsilon)$. Hence,

$$|A|_e < \sum \text{vol}(R_i(\epsilon)) = \epsilon$$

Hence $|A|_e = 0$, and therefore A is Lebesgue-measurable.

Let $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ be measure zero sets. Fix $\epsilon > 0$, and pick $U_i \supset \Gamma_i$ open such that $|U_i| < \epsilon/2^i$. Then $U = \cup_i U_i$ is an open set containing $\Gamma = \cup_i \Gamma_i$ and

$$|U| < \sum |U_i| = \epsilon$$

Hence, $|\Gamma| = 0$.

Finally, $[0, 1]$ is Lebesgue-measurable because it is closed, and $\mathbb{Q} \cap [0, 1]$ is Lebesgue-measurable because it is countable. Hence, $A = [0, 1] - \mathbb{Q}$ is Lebesgue-measurable, and

$$|A| = |[0, 1]| - |\mathbb{Q} \cap [0, 1]| = 1 - 0 = 1$$

because of the following lemma.

Lemma 1. If $B \subset A$ are measurable sets, then $|A - B| = |A| - |B|$

Proof. As A and B are measurable, so is $A - B = A \cap B^c$. As $(A - B) \cap B = \emptyset$, we have

$$|A - B| + |B| = |(A - B) \cup B| = |A|$$

□

Problem 2. Let $\Gamma_1, \Gamma_2, \dots$ be measurable sets such that $|\Gamma_i \cap \Gamma_j| = 0$. Let $\tau_i = \Gamma_i - (\cup_{j=1}^{i-1} \Gamma_j)$. Then τ_i are measurable, and $\tau_i \cap \tau_j = \emptyset$. Moreover,

$$|\tau_i| = |\Gamma_i| - \sum_{j=0}^{i-1} |\Gamma_i \cap \Gamma_j| = |\Gamma_i|$$

(by lemma above).

Hence,

$$|\cup_i \Gamma_i| = |\cup_i \tau_i| = \sum_i |\tau_i| = \sum_i |\Gamma_i|$$

Problem 3. Part (i): Note that each C_k is closed, as $C_{k+1} = C_k - \text{open set}$. Hence, $C = \cap_k C_k$ is closed as well. Also note that, by induction, C_n is the union of 2^n intervals, each of length 3^{-n} . Indeed, $C_0 = I$ is a single interval of length 1. To construct C_{k+1} from C_k , we replace each interval of length 3^{-k} by two intervals of length $1/3 \times 3^{-k}$, for a total of 2×2^k intervals.

Date: February 17, 2010.

Therefore, $|C_k| = (2/3)^k$. Hence, $0 \leq |C| \leq |C_k| = (2/3)^k$, for any k . We conclude that $|C| = 0$.

Part(ii): Let $\mathcal{A}$ be the set of $\alpha \in \{0, 1, 2\}^{\mathbb{N}}$ such that

- (1) $\alpha_0 \in \{0, 1\}$, and $\alpha_0 = 1$ only if $\alpha_k = 0$ for all $k > 0$
- (2) $\alpha_k \in \{0, 1\}$ for infinitely many k

Define a map $\alpha \mapsto \sum_{k=0}^{\infty} \alpha_k/3^k \in [0, 1]$. This map is surjective, because given any $t \in [0, 1]$, get the expansion $(\alpha_0, \alpha_1\alpha_2 \dots)$ of t in base 3 such that it does not end in a $222 \dots$ sequence. Then α is an element of $\mathcal{A}$, and it maps back to t . This map is injective as well, because representation in base 3 is unique for all but the numbers that admit finite expansion $(\alpha_0, \alpha_1\alpha_2 \dots \alpha_k)$. For these, there is a second expansion $(\alpha_0, \alpha_1 \dots, \alpha_{k-1}(\alpha_k - 1)222 \dots)$. Of these two possible representations, exactly one is an element of $\mathcal{A}$. Hence, we proved that this map is a bijection. Let $x \mapsto \alpha(x) \in \mathcal{A}$ denote the inverse map.

Let $\mathcal{A}_0$ denote the subset of $\mathcal{A}$ that $\alpha_k \neq 0$ for only finitely many values of k . Note that

$$\mathcal{A}_0 = \cup_{k \in \mathbb{N}} \{0, 1, 2\}^k$$

is a countable union of finite sets, hence it is countable.

Let $\mathcal{A}^* = \{0, 2\}^{\mathbb{N}}$ denote the set of α 's such that $\alpha_k \in \{0, 2\}$ for all k . By Cantor's diagonal argument, this set is uncountable. (That is, for any map $c : \mathbb{N} \rightarrow \{0, 2\}^{\mathbb{N}}$, the element $\alpha_k = 2 - c(k)_k$ is not in the image, so c cannot be surjective).

Hence, $\mathcal{A}' = \mathcal{A}^* - \mathcal{A}_0$ is an uncountable set.

Claim 1. A point x is in the interior of C_l if, and only if, $\alpha(x)_k \in \{0, 2\}$ for every $0 \leq k \leq l$, and $\alpha(x)_k \neq 0$ for some $k > l$.

By this claim, a point x is in the interior of all C_l , if, and only if, $\alpha(x)_k \in \{0, 2\}$ for all k , and $\alpha(x)_k \neq 0$ for infinitely many values of k . That is, $\alpha(\cap_l \text{int}(C_l)) = \mathcal{A}'$. As α is a bijection, and $\mathcal{A}'$ is uncountable, so is $\cap_l \text{int}(C_l)$. Therefore, the set $C = \cap C_l$ contains an uncountable subset. Therefore, C is uncountable as well. Hence, it is enough to prove the claim above.

To do this, note, by induction, that

$$C_n = \cup_m \left[\frac{m}{3^n}, \frac{m+1}{3^n} \right]$$

where the union runs over integers $0 \leq m \leq 3^n$ such that the base 3 representation of m contains only 0 and 2. Indeed, for $n = 0$ this is obvious. For the step case, it is enough to notice that we partition the interval $[\frac{m}{3^n}, \frac{m+1}{3^n}]$ into

$$\left[\frac{3m}{3^{n+1}}, \frac{3m+1}{3^{n+1}} \right] \cup \left[\frac{3m+2}{3^{n+1}}, \frac{3m+3}{3^{n+1}} \right]$$

Hence, the description above for C_n is proved, and so is the claim.

Problem 4. Let A be a compact set. Let $\mu(A) = \inf_{\mathcal{C} \text{ finite}} \Sigma(\mathcal{C})$.

It is clear that

$$|A| = \inf_{\mathcal{C} \text{ countable}} \Sigma(\mathcal{C}) \leq \inf_{\mathcal{C} \text{ finite}} \Sigma(\mathcal{C}) = \mu(A)$$

where the infimum runs over covers by rectangles. Conversely, fix $\epsilon > 0$, and get any countable cover $\mathcal{C} = \{R_1, R_2, \dots\}$. For each i , pick an open rectangle $R'_i \supset R_i$ such that

$vol(R_i) + \epsilon/2^i > vol(R'_i)$. So $R'_1, R'_2, \dots$ is an open cover of the compact set A , which can be refined to a finite subcover $\mathcal{C}' = \{R'_1, R'_2, \dots, R'_N\}$.

Hence,

$$\mu(A) = \inf_{\mathcal{C} \text{ finite}} \Sigma(\mathcal{C}) \leq \Sigma(\mathcal{C}') = \sum_{i=1}^N |R'_i| < \sum_{i=1}^{\infty} |R_i| + \epsilon/2^i = \Sigma(\mathcal{C}) + \epsilon$$

As this is true for all $\epsilon > 0$, we conclude

$$\mu(A) \leq \Sigma(\mathcal{C})$$

As this is valid for all countable covers $\mathcal{C}$, then $\mu(A) \leq |A|$. Hence, $\mu(A) = |A|$.

Problem 5. Let $\Gamma_1, \Gamma_2, \dots$ measurable sets, such that $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$. Let A be any set. On one hand,

$$|\cup_i (\Gamma_i) \cap A|_e \leq \sum_i |\Gamma_i \cap A|_e$$

since this is true in general.

On the other hand, given any open set $U \supset A \cap (\cup_i \Gamma_i)$,

$$\sum_i |\Gamma_i \cap A|_e \leq \sum_i |\Gamma_i \cap U|_e$$

But as U is open, and Γ_i measurable, $U \cap \Gamma_i$ is measurable, and

$$\sum_i |\Gamma_i \cap U|_e = \sum_i |\Gamma_i \cap U| = |\cup_i \Gamma_i \cap U| \leq |U|$$

Hence, $|U| \geq \sum_i |\Gamma_i \cap A|_e$ for all $U \supset A \cap (\cup_i \Gamma_i)$. Therefore,

$$|A \cap (\cup_i \Gamma_i)|_e = \inf(|U|) \geq \sum_i |\Gamma_i \cap A|_e$$

and we are done.