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5.a

5.a.1 A Lemma

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Suppose $m > 0$ and $0 < b < 1$. Then:

$$m^b - mb + b - 1 \leq 0$$

The expression is a C^∞ function of m on the positive real numbers, so this can be verified by taking derivatives.

Its first derivative is $bm^{b-1} - b$, so there is one critical point, at $m = 1$. The value of $m^b - mb + b - 1$ there is 0. The second derivative is $(b-1)bm^{b-2}$. Since $0 < b < 1$, this is negative for all $m > 0$; thus the value 0 at $m = 1$ is maximal.

5.a.2 Part (a) continued

If either $u = 0$ or $v = 0$, then $uv = 0$. Since the right side $\frac{u^p}{p} + \frac{v^q}{q}$ must be nonnegative given that all of u, v, p , and q are nonnegative, the inequality would hold.

So assume both u and v are strictly positive. Set $m = \frac{v^q}{u^p}$ and $b = \frac{1}{q}$ and apply the lemma above:

$$\begin{aligned} \frac{v^q}{u^p} - \frac{v^q}{u^p} \frac{1}{q} + \frac{1}{q} - 1 &\leq 0 \\ v u^{p(1-\frac{1}{q})} - \frac{v^q}{q} + \left(\frac{1}{q} - 1\right) u^p &\leq 0 \\ v u^{\frac{p}{q}} - \frac{v^q}{q} + \frac{u^p}{p} &\leq 0 \\ uv &\leq \frac{u^p}{p} + \frac{v^q}{q} \end{aligned}$$

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5.b

If either integral on the right hand side is infinite, the inequality is tautological. If either is zero, then theorem (1.2.10) implies that either $f \equiv 0$ or $g \equiv 0$, making $fg \equiv 0$ and $\int_J fg \, d\psi$ zero. Thus the inequality holds if either is zero.

Having taken care of these cases, assume now that both integrals on the right hand side are finite and nonzero. Then by dividing through, we may replace f and g by $(\int_J f^p \, d\psi)^{-1/p} f$ and $(\int_J g^q \, d\psi)^{-1/q} g$ so that both integrals on the right hand side are equal to 1.

From part (a), we have the inequality:

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$