

over the remaining intervals and is at most $\frac{\epsilon}{2M} \Delta_J \psi < \frac{\epsilon}{2}$. Thus by restricting the mesh size we can keep each cover's possible Riemann sums arbitrarily close together.

Suppose now that $\mathcal{C}'$ is a refinement of $\mathcal{C}$, with corresponding choice functions ξ' and ξ . Suppose $I \in \mathcal{C}$ splits into $I_i \in \mathcal{C}'$ and consider the error:

$$\begin{aligned} & \phi(\xi(I)) \Delta_I \psi - \sum_i \phi(\xi'(I_i)) \Delta_{I_i} \psi \\ &= \sum_i (\phi(\xi(I)) - \phi(\xi'(I_i))) \Delta_{I_i} \psi \end{aligned}$$

The crucial observation here is that the $\phi(\xi'(I_i))$ all lie within the range of possibilities for $\phi(\xi(I))$, and so the error between a cover and its refinement is less than the error between some two choice functions of the coarser cover.

Now if we're given $\epsilon > 0$, fix a mesh size limit $\delta > 0$ such that for any two choice functions over the same cover $\mathcal{C}$ with $\|\mathcal{C}\| < \delta$, the difference in the resulting Riemann sums is less than $\frac{\epsilon}{2}$. Given two covers $\mathcal{C}_1$ and $\mathcal{C}_2$ with mesh size less than δ , let $\mathcal{C}$ be their common refinement. We then apply the triangle inequality:

$$\begin{aligned} |\mathcal{R}(\phi|\psi; \mathcal{C}_1; \xi_1) - \mathcal{R}(\phi|\psi; \mathcal{C}; \xi'_1)| &< \frac{\epsilon}{2} \\ |\mathcal{R}(\phi|\psi; \mathcal{C}_2; \xi_2) - \mathcal{R}(\phi|\psi; \mathcal{C}; \xi'_2)| &< \frac{\epsilon}{2} \\ |\mathcal{R}(\phi|\psi; \mathcal{C}_1; \xi_1) - \mathcal{R}(\phi|\psi; \mathcal{C}_2; \xi_2)| &< \epsilon \end{aligned}$$

Thus ϕ is ψ -Riemann integrable.

20/20 4 Composition with a continuous map

Since ϕ is continuous and defined on a closed interval, its image is (closed and) bounded; thus $\phi \circ f$ is bounded. Furthermore, ϕ is uniformly continuous; that is, given $\epsilon > 0$ there exists $\gamma > 0$ such that $|x - y| < \gamma$ implies $|\phi(x) - \phi(y)| < \epsilon$.

Let $\gamma' > 0$ be smaller than both γ and ϵ , and apply the previous problem to f . That is, take $\delta > 0$ such that if $\|\mathcal{C}\| < \delta$, then:

$$\sum_{I \in \mathcal{C}: \sup_I f - \inf_I f \geq \gamma'} \Delta_I \psi < \gamma'$$

By uniform continuity of ϕ , for $I \in \mathcal{C}$:

$$\sup_I \phi \circ f - \inf_I \phi \circ f \geq \epsilon \Rightarrow \sup_I f - \inf_I f \geq \gamma'$$

Hence:

$$\sum_{I \in \mathcal{C}: \sup_I \phi \circ f - \inf_I \phi \circ f \geq \epsilon} \Delta_I \psi < \gamma' < \epsilon$$

Using the previous problem again, we conclude that $\phi \circ f$ is ψ -Riemann integrable.