

The integrals, if they exist, must be the supremums of these; hence we have an inequality of integrals as well. By symmetry the same applies to g , so:

$$(R) \int_J (f \vee g) \, dx \geq \left(\int_J f \, dx \right) \left(\int_J g \, dx \right)$$

The analogous result for $f \wedge g$ follows by flipping the signs of f and g ; and if $f \leq g$ this reduces to the statement that:

$$(R) \int_J f \, dx \leq (R) \int_J g \, dx$$

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If ϕ is ψ -Riemann integrable, then there exists a number A such that for any $\epsilon > 0$, a mesh size limit δ can be chosen so that for all admissible covers with $|\mathcal{C}| < \delta$, we have:

$$|\mathcal{R}(\phi|\psi; \mathcal{C}; \xi) - A| < \frac{\epsilon^2}{4}$$

In particular this implies that for any two choice functions, the resulting Riemann sums differ by no more than $\frac{\epsilon^2}{2}$. Consider now the sum:

$$\sum_{I \in \mathcal{C}: \sup_I \phi - \inf_I \phi \geq \epsilon} \frac{\epsilon}{2} \Delta_I \psi$$

Since on these intervals the supremum and infimum of ϕ differ by at least ϵ , some two choice functions exist that differ by more than $\frac{\epsilon}{2}$ on these intervals. Hence for appropriate ξ_1 and ξ_2 , this is less than:

$$\sum_{I \in \mathcal{C}: \sup_I \phi - \inf_I \phi \geq \epsilon} (\phi(\xi_1(I)) - \phi(\xi_2(I))) \Delta_I \psi$$

This is less than the sum over all $I \in \mathcal{C}$, which is merely the difference in the Riemann sums and is therefore less than $\frac{\epsilon^2}{2}$. Dividing through by $\frac{\epsilon}{2}$, we have:

$$\sum_{I \in \mathcal{C}: \sup_I \phi - \inf_I \phi \geq \epsilon} \Delta_I \psi < \epsilon$$

To prove the other implication, suppose this inequality holds for some $\frac{\epsilon}{2M} > 0$, where M is chosen to be greater than both 1 and $\Delta_J \psi$. Then in estimating the difference between two Riemann sums over the same $\mathcal{C}$, we split the following sum in two:

$$\sum_{I \in \mathcal{C}} (\phi(\xi_1(I)) - \phi(\xi_2(I))) \Delta_I \psi$$

One sum will be taken over intervals I for which $\sup_I \phi - \inf_I \phi \geq \frac{\epsilon}{2M}$. This, by the inequality we assume, is less than $\frac{\epsilon}{2M}$. The other sum will be taken