

2.b Combining functions

First we will show that given Riemann integrable f , its absolute value $|f|$ is Riemann integrable. Given $\epsilon > 0$, take $\delta > 0$ such that Riemann upper and lower sums of f over $\mathcal{C}$ with mesh size less than δ lie within ϵ of each other. For such a cover $\mathcal{C}$, consider the difference:

$$\mathcal{U}(|f|; \mathcal{C}) - \mathcal{L}(|f|; \mathcal{C}) = \sum_{I \in \mathcal{C}} \left(\sup_I |f| - \inf_I |f| \right) \text{vol}(I)$$

Fix a rectangle I . If f has the same sign over all of I , then:

$$\sup_I (-f) - \inf_I (-f) = - \left(\inf_I f - \sup_I f \right) = \sup_I f - \inf_I f$$

If f does not have the same sign over all of I , then:

$$\sup_I |f| - \inf_I |f| \leq \sup_I |f| \leq \sup_I f - \inf_I f$$

Hence if we sum over all $I \in \mathcal{C}$, we have:

$$\sum_{I \in \mathcal{C}} \left(\sup_I |f| - \inf_I |f| \right) \text{vol}(I) \leq \sum_{I \in \mathcal{C}} \left(\sup_I f - \inf_I f \right) \text{vol}(I)$$

This is just equal to the difference between the Riemann upper and lower sums of f , which is less than ϵ by our choice of $\mathcal{C}$. By choosing ϵ to be arbitrarily small we thus find upper and lower sums arbitrarily close to each other. Therefore (1.1.4) is satisfied:

$$\sup_{\mathcal{C}} \mathcal{L}(f; \mathcal{C}) \geq \inf_{\mathcal{C}} \mathcal{U}(f; \mathcal{C})$$

Thus $|f|$ is Riemann integrable if f is.

Linearity of the integral follows from linearity of Riemann sums; since Riemann sums are taken over finite covers, we are free to rearrange them:

$$\sum (\alpha f(\xi(I)) + \beta g(\xi(I))) \text{vol}(I) = \alpha \left(\sum f(\xi(I)) \right) + \beta \left(\sum g(\xi(I)) \right)$$

Let A and B be the Riemann integrals of f and g , respectively. Apply the triangle inequality:

$$\begin{aligned} & |\mathcal{R}(\alpha f + \beta g; \mathcal{C}; \xi) - \alpha A + \beta B| \\ &= |\alpha \mathcal{R}(f; \mathcal{C}; \xi) + \beta \mathcal{R}(g; \mathcal{C}; \xi) - \alpha A + \beta B| \\ &\leq |\alpha| |\mathcal{R}(f; \mathcal{C}; \xi) - A| + |\beta| |\mathcal{R}(g; \mathcal{C}; \xi) - B| \end{aligned}$$

For $\epsilon > 0$, if we take the mesh size so that $|\mathcal{R}(f; \mathcal{C}; \xi) - A| < \frac{\epsilon}{2|\alpha|}$ and $|\mathcal{R}(g; \mathcal{C}; \xi) - B| < \frac{\epsilon}{2|\beta|}$, this will be less than ϵ . It follows from this that the integral of $\alpha f + \beta g$ is merely $\alpha A + \beta B$.

Now we can write $f \vee g$ as $\frac{f+g+|f-g|}{2}$ and see from the results above that it is Riemann integrable. Since $f \leq f \vee g$, for any $\mathcal{C}$ we have the inequality of Riemann lower sums:

$$\mathcal{L}(f; \mathcal{C}) \leq \mathcal{L}(f \vee g; \mathcal{C})$$