

We subtract $\phi(a_m)d_m$ and rearrange terms:

$$\begin{aligned} & \phi^+(\psi^+ - \psi^0) + \phi^-(\psi^0 - \psi^-) - \phi^0(\psi^+ - \psi^-) \\ &= \psi^+(\phi^+ - \phi^0) + \psi^-(\phi^0 - \phi^-) + \psi^0(\phi^- - \phi^+) \end{aligned}$$

By our choice of δ , each of the differences in values of ϕ is less than $2r\epsilon$. Having taken M as a bound on ψ , the absolute value of this expression is less than $6Mr\epsilon = \frac{\epsilon}{2n}$.

At the endpoints a_0 and a_n the arithmetic works out similarly; we end up with one of the following:

$$\begin{aligned} & \psi^+(\phi^+ - \phi^0) + \psi^0(\phi^0 - \phi^+) \\ & \psi^-(\phi^0 - \phi^-) + \psi^0(\phi^- - \phi^0) \end{aligned}$$

Thus each endpoint also contributes less than $\frac{\epsilon}{2n}$. Adding everything together for all the a_m , we see:

$$\left| \mathcal{R}(\phi|\psi; \mathcal{C}'; \xi') - \sum \phi(a_m)d_m \right| < \frac{\epsilon(n+1)}{2n}$$

Now what remains to be done is reconciling the Riemann sums over $\mathcal{C}$ and $\mathcal{C}'$. At each point a_m , if the interval $I \in \mathcal{C}$ containing a_m did not have a_m as an endpoint, it became two intervals in $\mathcal{C}'$. Fix $\xi \in \Xi(\mathcal{C})$ and choose $\xi' \in \Xi(\mathcal{C}')$ by setting $\xi'(I) = \xi(I)$ for I in both $\mathcal{C}$ and $\mathcal{C}'$. Then the Riemann sums differ only over intervals that were split.

At each point a_m lying in a split interval $I \in \mathcal{C}$, the relevant term in $\mathcal{R}(\phi|\psi; \mathcal{C}; \xi)$ is $\phi(\xi(I))(\psi(a_m+) - \psi(a_m-))$; and the terms in $\mathcal{R}(\phi|\psi; \mathcal{C}'; \xi')$ are as before. Adopting the same abbreviations as before, we subtract and rearrange terms:

$$\begin{aligned} & \phi^+(\psi^+ - \psi^0) + \phi^-(\psi^0 - \psi^-) - \phi(\xi(I))(\psi^+ - \psi^-) \\ &= \psi^+(\phi(\xi(I)) - \phi^0) + \psi^-(\phi^0 - \phi(\xi(I))) + \psi^0(\phi^- - \phi^+) \end{aligned}$$

Again since all the points at which ϕ is evaluated lie within δ of a_m , the absolute value of this expression is less than $6Mr\epsilon = \frac{\epsilon}{2n}$. Since the intervals at the endpoints a_0 and a_n couldn't have been split, there are at most $n-1$ points which will contribute something to this difference. Hence:

$$\left| \mathcal{R}(\phi|\psi; \mathcal{C}'; \xi') - \mathcal{R}(\phi|\psi; \mathcal{C}; \xi) \right| < \frac{\epsilon(n-1)}{2n}$$

Combining these inequalities using the triangle inequality, we finally have:

$$\left| \mathcal{R}(\phi|\psi; \mathcal{C}; \xi) - \sum \phi(a_m)d_m \right| < \frac{\epsilon(n+1)}{2n} + \frac{\epsilon(n-1)}{2n} = \epsilon$$