

18.125 Assignment 1

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1 Riemann-Stieltjes Integration with respect to piecewise constant functions

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We will proceed by estimating the error:

$$\left| \mathcal{R}(\phi|\psi; \mathcal{C}; \xi) - \sum_{m=0}^n \phi(a_m) d\psi \right|$$

With finitely many intervals (a_m, a_{m+1}) , ψ takes on finitely many values, so it is bounded. Take $M > 0$ such that $M > |\psi(x)|$ for all $x \in J$, and let $r = \frac{1}{12nM}$.

Now suppose we're given $\epsilon > 0$. Since ϕ is continuous, we can choose $\delta_m > 0$ at each point a_m such that $|x - a_m| < \delta_m$ implies $|\phi(x) - \phi(a_m)| < r\epsilon$. From the triangle inequality we have that for any x and y within δ_m of a_m , $|\phi(x) - \phi(y)| < 2r\epsilon$. Since there are finitely many points a_m , there is a minimal δ_m , which we'll call δ' .

All of the a_m are distinct, so the set of distances $|a_i - a_j|$ consists of positive numbers. Since there are finitely many a_m , this is a finite set, so it has a positive minimum, which we'll call δ'' .

Now take δ to be strictly less than both δ' and δ'' . Let $\mathcal{C}$ be an admissible cover of J with $|\mathcal{C}| < \delta$, and let $\mathcal{C}'$ be the refinement obtained by replacing $[x, y]$ with $[x, a_m]$ and $[a_m, y]$ if $a_m \in (x, y)$. This is still finite (since the set of a_m is finite), non-overlapping, and exact; and we still have $|\mathcal{C}'| < \delta$. Let ξ' be a choice function on $\mathcal{C}'$. We will try to simplify the Riemann sum:

$$\mathcal{R}(\phi|\psi; \mathcal{C}'; \xi') = \sum_{I \in \mathcal{C}'} \phi(\xi'(I)) \Delta_I \psi$$

By our choice of δ , each $I \in \mathcal{C}'$ contains at most one of the points a_m ; and by the construction of $\mathcal{C}'$ from $\mathcal{C}$, such a point a_m must lie at an endpoint of I . We can discard the intervals I not containing a point a_m , since $\Delta_I \psi$ would be zero for such I .

For the moment we restrict our attention to a single point a_m . Let $\phi^0 = \phi(a_m)$, and let ϕ^- and ϕ^+ be the values of $\phi \circ \xi'$ evaluated at the intervals to the left and right of a_m , respectively. Let $\psi^0 = \psi(a_m)$, $\psi^+ = \psi(a_m+)$, and $\psi^- = \psi(a_m-)$.

For points other than a_0 and a_n , the relevant terms in the Riemann sum are:

$$\phi^+(\psi^+ - \psi^0) + \phi^-(\psi^0 - \psi^-)$$