

Conversely, suppose $\sum_{n=1}^{\infty} \delta_n^2 < \infty$. We will show that every sequence in S_1 has a convergent subsequence, which implies S_1 is compact. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in S_1 . Taking inner products with u_1 gives a sequence $\{(f_i, u_1)\}$. Because $\{(f_i, u_1)\}$ lies in the compact set $\{z \in \mathbb{C} : |z| \leq \delta_1\}$, it has a convergent subsequence. Let $\{f_i^{(1)}\}$ be the corresponding subsequence of $\{f_i\}$. In a similar fashion, for each $k > 0$, inductively choose a subsequence $\{f_i^{(k+1)}\}$ of $\{f_i^{(k)}\}$ such that $\{(f_i^{(k+1)}, u_{k+1})\}$ converges.

Let $g_i = f_i^{(i)}$. We claim that the g_i form a Cauchy sequence in S_1 . Let $\epsilon > 0$, and choose N such that $\sum_{n>N} \delta_n^2 < \epsilon/2$. For each k , there exists an M_k such that $|(g_m - g_n, u_i)| < \sqrt{\epsilon/2N}$ for $n > m > M_k$. Then, if $n > m > \max(M_1, \dots, M_k)$, we have

$$\|g_n - g_m\|^2 = \sum_{i=1}^{\infty} |(g_n - g_m, u_i)|^2 < N \cdot \frac{\epsilon}{2N} + \sum_{i=N+1}^{\infty} |(g_n - g_m, u_i)|^2 = \epsilon$$

Hence g_n is Cauchy, and therefore converges to some $g \in H$. Because $(g_n, u_i) \rightarrow (g, u_i)$ for all i and $|(g_n, u_i)| \leq \delta_i$ we conclude $|(g, u_i)| \leq \delta_i$, which implies $g \in S_1$. So any sequence $\{f_i\}$ in S_1 has a convergent subsequence $\{g_i\}$, showing that S_1 is compact.

20/20 10.3. Let $L : L^1([0, 1]) \rightarrow \mathbb{C}$ be the functional $Lf = \int_0^1 f dx$. This map is linear, and it is bounded, since $|Lf| = |\int f dx| \leq \int |f| dx = \|f\|$. Hence L is continuous, and since $M = L^{-1}(\{1\})$, it follows that M is closed.

M is certainly convex: if $f, g \in M$ and $0 \leq \lambda \leq 1$, then $L(\lambda f + (1-\lambda)g) = \lambda Lf + (1-\lambda)Lg = 1$.

Finally, if $f \in M$, we have $\|f\| \geq |Lf| = 1$. So all elements in M have norm at least 1. However, infinitely many $f \in M$ have norm exactly 1; for instance, the scaled characteristic functions $f_a = a^{-1} \cdot 1_{[0, a]}$ lie in M and have norm 1 for all $a \in (0, 1]$.

20/20 10.4. From linear algebra, we know $X \times Y$ is a vector space, so it suffices to show that $\|(x, y)\|$ is a norm and that the associated normed space is complete. First, we verify $\|(x, y)\|$ is a norm. For all $(x, y) \in X \oplus Y$ we have $\|(x, y)\| = \|x\|_X + \|y\|_Y \geq 0$, and $\|(x, y)\| = 0$ implies $\|x\|_X = \|y\|_Y = 0$, so that $(x, y) = 0$. Scaling is OK, as $\|\alpha(x, y)\| = \|(\alpha x, \alpha y)\| = \|\alpha x\|_X + \|\alpha y\|_Y = |\alpha| \cdot (\|x\|_X + \|y\|_Y) = |\alpha| \cdot \|(x, y)\|$. Finally, we have the triangle inequality:

$$\|(x, y) + (a, b)\| = \|x + a\|_X + \|y + b\|_Y \leq \|x\|_X + \|a\|_X + \|y\|_Y + \|b\|_Y = \|(x, y)\| + \|(a, b)\|.$$

So we indeed have a norm.

For completeness, suppose that $\{(x_n, y_n)\}_{n=1}^{\infty}$ is Cauchy in $X \oplus Y$. Since $\|(x_m, y_m) - (x_n, y_n)\| \leq \|x_m - x_n\|_X$, the sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy in X , and hence converges to some $x \in X$. Similarly, $\{y_n\}_{n=1}^{\infty}$ converges to some $y \in Y$. But $\|(x_n, y_n) - (x, y)\| = \|x_n - x\|_X + \|y_n - y\|_Y$, which converges to 0 as $n \rightarrow \infty$. So $(x_n, y_n) \rightarrow (x, y) \in X \oplus Y$, and we have shown $X \oplus Y$ is complete.

For the closed graph theorem, suppose $x \mapsto \Lambda x$ is continuous. Then the map $A : (x, y) \rightarrow y - \Lambda x$ is continuous, implying that $A^{-1}(\{0\}) = \Gamma(\Lambda)$ is closed.

Conversely, suppose $\Gamma(\Lambda)$ is closed. Now $\Gamma(\Lambda)$ is a vector subspace of $X \times Y$: if $(x, \Lambda x), (y, \Lambda y) \in \Gamma(\Lambda)$ and $\alpha, \beta \in \mathbb{C}$, then

$$\alpha(x, \Lambda(x)) + \beta(y, \Lambda(y)) = (\alpha x + \beta y, \Lambda(\alpha x + \beta y)) \in \Gamma(\Lambda)$$