

20/20 10.1. We start by showing the finite subsets $\{u_1, \dots, u_N\}$ are orthonormal and satisfy $\{u_1, \dots, u_N\} = \{x_1, \dots, x_N\}$. Certainly $\{u_1\}$ is an orthonormal set. Now suppose $\{u_1, \dots, u_{N-1}\}$ is orthonormal. Then v_N is nonzero, since $x_1, \dots, x_N$ are linearly independent, so u_N is defined and $\|u_N\| = \|v_N\| / \|v_N\| = 1$. Now, for any $m < N$,

$$\|v_N\| \cdot (u_m, u_N) = (u_m, v_N) = (x_N, u_m) - \sum_{i=1}^{N-1} (x_N, u_i)(u_i, u_m) = (x_N, u_m) - (x_N, u_m)(u_m, u_m) = 0$$

So $\{u_1, \dots, u_N\}$ is orthonormal; by induction this is true for all N . At the same time, we have shown that $\{u_n\}_{n=1}^{\infty}$ is an orthonormal set, since any two vectors u_m, u_n of $\{u_n\}_{n=1}^{\infty}$ belong to a finite subset of $\{u_n\}_{n=1}^{\infty}$.

Next, suppose that $\text{span}\{u_1, \dots, u_{N-1}\} = \text{span}\{x_1, \dots, x_{N-1}\}$. We know $x_N = v_N + \sum_{i=1}^{N-1} (x_N, u_i)u_i \in \text{span}\{u_1, \dots, u_N\}$, and hence $\text{span}\{u_1, \dots, u_N\} \supseteq \text{span}\{x_1, \dots, x_N\}$. Because the u_i are linear combinations of the x_i , the opposite inclusion also holds. By induction, $\text{span}\{u_1, \dots, u_N\} = \text{span}\{x_1, \dots, x_N\}$ for all N .

Lastly, suppose H is separable, and let $\{y_r\}$ be a countable dense subset of H . Construct a linearly independent subset $X = \{x_n\} \subseteq \{y_r\}$ as follows. Let $X_0 = \emptyset$, and define X_n inductively by setting $X_n = X_{n-1}$ if $y_n \in \text{span}\{y_1, \dots, y_{n-1}\}$ and $X_n = X_{n-1} \cup \{y_n\}$ otherwise. Then set $X = \bigcup X_n$. Certainly $\text{span}\{x_n\}_{n=1}^{\infty} \subseteq \text{span}\{y_r\}_{r=1}^{\infty}$, and we have equality because if $y_r \notin \text{span}\{x_n\}$ we would have $y_r \in X_r$, a contradiction.

Using Gram-Schmidt we obtain an orthonormal set $\{u_n\}$ from $\{x_n\}$. To see this is a basis for H , suppose $a \in H$ but $(a, u_i) = 0$ for all i . Then $(a, x_i) = 0$ also, and hence $(a, x) = 0$ for all $x \in \text{span}\{x_n\} = \text{span}\{y_r\}$. But $\text{span}\{y_r\}$ is dense in H , so for any $\epsilon > 0$, there exists some $b \in \text{span}\{y_r\}$ with $\|a - b\| < \epsilon$. So

$$\|a\|^2 = (a, a - b) + (a, b) = (a, a - b) + 0 \leq \|a\| \|a - b\| < \epsilon \|a\|$$

implying $\|a\| < \epsilon$. Hence $a = 0$, showing $\{u_n\}$ is an orthonormal basis.

20/20 10.2. (a) If $m \neq n$, then $\|u_m - u_n\|^2 = (u_m, u_m) - 2\text{Re}(u_m, u_n) + (u_n, u_n) = 2$. So S is discrete and hence closed. It is certainly bounded, as $\|u_n\| = 1$ for all n . To see S is not compact, take the covering $\{U_n\}_{n=1}^{\infty}$ of open sets $U_n = B(1, u_n)$. Each U_n contains the corresponding u_n , but no other u_m . As a result, $\{U_n\}$ can have no finite subcover, so S_1 is not compact.

It follows that the closed unit ball cannot be compact; otherwise S would also be compact since S is a closed subset of the unit ball. If U is a nonempty open set, it contains a closed ball of radius ϵ for sufficiently small ϵ , which is homeomorphic to the closed unit ball and therefore not compact. Hence $\bar{U}$ cannot be compact. This shows that H has no nontrivial precompact subsets, implying H is not locally compact.

20/20 (b) If $\sum_{n=1}^{\infty} \delta_n^2 = \infty$, then S_1 is not bounded. To see this, let $x_N = \sum_{n=1}^N \delta_n u_n \in S_1$ for $N = 1, 2, \dots$. Then $\|x_N\|^2 = \sum_{n=1}^N \delta_n^2 \rightarrow \infty$ as $N \rightarrow \infty$. Since S_1 is not bounded, it is certainly not compact.