

MIDTERM FOR 18.125, SPRING 2010
THURSDAY, MARCH 18

PRINTED Name: _____

- Write your name on EVERY page. Do all your work on these pages. No books or notes may be used.
- (1) (a) Prove $H_0 = \{y \in \mathbb{R}^N : y_1 = 0\}$ has Lebesgue measure 0.
 (b) Prove if $H \subset \mathbb{R}^N$ is a hyperplane, H has Lebesgue measure 0. (H a hyperplane means there exists $c \in \mathbb{R}^N$ and $\ell \in \mathbb{R}^N \setminus \{0\}$ so that $H = \{y \in \mathbb{R}^N : (y - c) \cdot \ell = 0\}$.)

Solution to a: Let μ denote Lebesgue measure and let

$$A_{n,\epsilon} = \left[-\frac{\epsilon}{2^{N+n}n^{N-1}}, \frac{\epsilon}{2^{N+n}n^{N-1}} \right] \times [-n, n] \cdots \times [-n, n] \subset \mathbb{R}^{N-1}.$$

Then $H_0 \subset \cup_{n=1}^{\infty} A_{n,\epsilon}$ for every $\epsilon > 0$, so it follows that

$$\mu(H_0) \leq \sum_{n=1}^{\infty} \mu(A_{n,\epsilon}) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

for every $\epsilon > 0$. Hence $\mu(H_0) = 0$.

Solution to b: Let $H' = \{y \in \mathbb{R}^N : y \cdot \ell = 0\}$. Then $\mu(H') = \mu(H)$ since H' is obtained from H by a translation and Lebesgue measure is translation invariant. There exists an orthogonal linear transformation corresponding to the matrix A mapping H_0 onto H' (just choose an orthonormal basis of H' and complete it to $\mathbb{R}^N$ with the normal vector $\ell/|\ell|$). We have $|\det(A)| = 1$, and $\mu(H') = \mu(H_0) \cdot |\det(A)| = 0$, so $\mu(H) = 0$.

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- (2) Let $(E, \mathcal{B}, \mu)$ be a measure space and assume $\mu(E) < \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on E , and assume there exists a function f so that $f_n \rightarrow f$ μ -a.e. on E . Prove for any $\epsilon > 0$, there exists a measurable set $A \subset E$ and a number N such that

(i) $\mu(A) < \epsilon$, and

(ii) for all $n \geq N$ and all $x \in E \setminus A$, $|f_n(x) - f(x)| < \epsilon$.

(Hint: Consider a sequence of sets where the second conclusion does not hold, that is look at the sets $G_n = \{x \in E : |f_n(x) - f(x)| \geq \epsilon\}$.)

Solution:

Fix $\epsilon > 0$. Let $G_n = \{x \in E : |f_n(x) - f(x)| \geq \epsilon\}$ and let $H_n = \cup_{m \geq n} G_m$. If $x \in H_n$ for infinitely many n , then $f_n(x)$ does not converge to $f(x)$. Let $H = \cap_{n=1}^{\infty} H_n$, so that $H_n \searrow H$. Since $f_n \rightarrow f$ μ -a.e. on E , it follows that $\mu(H) = \mu(\cap_{n=1}^{\infty} H_n) = 0$. Since $H_n \searrow H$ and $\mu(H_1) \leq \mu(E) < \infty$, it follows that $\lim_{n \rightarrow \infty} \mu(H_n) = \mu(H) = 0$. Thus we can choose N so that $\mu(H_N) < \epsilon$. Letting $A = H_N$ it follows that A satisfies (i). Then for $n \geq N$ and $x \in E \setminus A$, we have that $|f_n(x) - f(x)| < \epsilon$, so (ii) is satisfied.

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- (3) Let $(E, \mathcal{B}, \mu)$ be a measure space, and suppose $f : E \rightarrow [0, \infty]$ is an integrable function such that

$$\int f d\mu = c$$

for some $c \in (0, \infty)$. Prove for $0 < \alpha < \infty$,

$$\lim_{n \rightarrow \infty} \int n \log[1 + (f/n)^\alpha] d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1, \\ c & \text{if } \alpha = 1, \\ 0 & \text{if } 1 < \alpha < \infty. \end{cases}$$

(*Hint:* Consider $\alpha \geq 1$ and $\alpha < 1$ separately. Then consider $\{f > n\}$ and $\{f \leq n\}$ and use an appropriate Taylor approximation.)

Solution: As we discussed in lecture, there were several ways to solve this problem. This solution is the one I had in mind for the hint.

Case 1: $\alpha \geq 1$. The idea is to use Dominated Convergence Theorem, but we need to find an appropriate *integrable* dominant. On $\{f > n\}$, $(f/n) > 1$ implies we can expand

$$\begin{aligned} n \log(1 + (f/n)^\alpha) &= n (\log((f/n)^\alpha) + \log(1 + (f/n)^{-\alpha})) \\ &= n\alpha \log(f/n) + n(f/n)^{-\alpha} + nE_1, \end{aligned}$$

where $|E_1| \leq (f/n)^{-2\alpha}/2$ by the alternating series remainder estimate. We have $f/n > 1$ and $\log$ is monotone increasing which implies $\log(f/n) \leq f/n$. Also, since $f/n > 1$, we have $n(f/n)^{-\alpha} \leq n < f$ and similarly for E_1 , so that

$$n \log(1 + (f/n)^\alpha) \leq n\alpha(f/n) + 2n \leq (\alpha + 2)f.$$

Now if $\{f \leq n\}$, we can expand

$$n \log(1 + (f/n)^\alpha) = n((f/n)^\alpha + E_2),$$

where $|E_2| \leq (f/n)^{2\alpha}/2$ by alternating series. Since $\alpha \geq 1$ and $f/n \leq 1$, $(f/n)^\alpha \leq f/n$. Hence we again have

$$n \log(1 + (f/n)^\alpha) \leq f \leq (\alpha + 2)f.$$

That means the sequence of functions

$$g_n = n \log(1 + (f/n)^\alpha)$$

is dominated by the integrable function $(\alpha + 2)f$ on all of E . Then the Dominated Convergence Theorem implies

$$\lim_{n \rightarrow \infty} \int g_n d\mu = \int \lim_{n \rightarrow \infty} g_n d\mu.$$

To compute the limit of g_n , we recall if $a \in [0, \infty]$, then

$$\lim_{n \rightarrow \infty} (1 + (a/n)^\alpha)^n = \begin{cases} e^a, & \text{if } \alpha = 1, \\ 1, & \text{if } \alpha > 1, \\ \infty, & \text{if } \alpha < 1. \end{cases}$$

Hence since $\{f = \infty\}$ has μ -measure 0,

$$\lim_{n \rightarrow \infty} g_n = \begin{cases} f, & \text{if } \alpha = 1, \\ 0, & \text{if } \alpha > 1, \\ \infty, & \text{if } \alpha < 1. \end{cases} \quad \mu\text{-a.e.}$$

This proves the formula for $\alpha \geq 1$.

Case 2: For $\alpha < 1$, we apply Fatou's Lemma and this limit with $\alpha < 1$:

$$\lim_{n \rightarrow \infty} \int g_n d\mu \geq \int \liminf_{n \rightarrow \infty} g_n d\mu = \infty.$$