

HOMEWORK 9 FOR 18.125, SPRING 2010
DUE THURSDAY, APRIL 29 AT THE BEGINNING OF
LECTURE.

NOTE: This homework is shorter because of the holiday.

HW9.1 Let H be a Hilbert space. If $M \subset H$ is a closed subspace, prove that $(M^\perp)^\perp = M$. Is the same true of M is not closed? What can you say when M is not closed?

HW9.2 This exercise is to help clarify subspaces and closed subspaces of a Hilbert space.

(a) Let $H = \mathbb{C}^N$ for some N , and show that any subspace of H is closed.

(b) Let $X = \mathbb{R}$ with Lebesgue measure, and let

$$H = \left\{ f \in L^2(\mathbb{R}) : \left\| \frac{d}{dx} f \right\|_{L^2} < \infty \right\}$$

equipped with the inner product

$$\langle f, g \rangle_H = \langle f, g \rangle_{L^2} + \left\langle \frac{d}{dx} f, \frac{d}{dx} g \right\rangle_{L^2}.$$

Show $H \subset L^2$ is a subspace. Show H is a Hilbert space, and hence a closed subspace of L^2 . Show that the function $f(x) = e^{-|x|}$ is not differentiable at 0 but nevertheless $f \in H$; that is, there are functions in H which are not differentiable in the usual sense. (*Hint:* To show completeness, consider difference quotients and use that Lebesgue measure is translation invariant.)

(c) Let $H = L^2((-1, 1))$ with Lebesgue measure. Let $M \subset H$ be the set

$$M = \{f \in L^2((-1, 1)) \cap C((-1, 1))\}.$$

Show that M is a subspace of $L^2((-1, 1))$, and give an example to show M is not closed.

HW9.3 Let H be a Hilbert space, and let $L : H \rightarrow \mathbb{C}$ be a continuous linear functional on H . Let $M \subset H$ be the closed subspace

$$M = \{x \in H : Lx = 0\},$$

and assume $M \neq H$. Show that $M^\perp$ has dimension 1.