

HOMEWORK 8 FOR 18.125, SPRING 2010
DUE THURSDAY, APRIL 22 AT THE BEGINNING OF
LECTURE.

HW8.1 Suppose $f : X \rightarrow \mathbb{C}$ is a measurable function, and μ is a non-negative measure on X . For $0 < p < \infty$, set

$$\phi(p) = \|f\|_{L^p(\mu)}^p.$$

Assume $\|f\|_{L^\infty(\mu)} > 0$, and set $E = \{p : \phi(p) < \infty\}$.

a) Prove if $r < p < s$ and $r, s \in E$ then $p \in E$ (in other words, E is connected).

b) Prove that $\log(\phi)$ is a convex function on the interior of E and that ϕ is continuous on E .

c) Is E necessarily open? Closed? Can E consist of only one point? Can E be any connected subset of $(0, \infty)$?

d) If $r < p < s$, prove that $\|f\|_{L^p} \leq \max\{\|f\|_{L^r}, \|f\|_{L^s}\}$. Show this implies the inclusion $L^r \cap L^s \subset L^p$.

e) Assume $\|f\|_{L^r} < \infty$ for some $r < \infty$ and prove

$$\|f\|_{L^p} \rightarrow \|f\|_{L^\infty}$$

as $p \rightarrow \infty$.

HW8.2 In addition to the assumptions of HW8.1, assume $\mu(X) = 1$.

a) Prove that $0 < r < s \leq \infty$ implies $\|f\|_{L^r} \leq \|f\|_{L^s}$.

b) Prove that $L^r(\mu) \supset L^s(\mu)$ if $0 < r < s$. Under what conditions are these spaces equal?

c) Assume $\|f\|_{L^r} < \infty$ for some $r > 0$ and prove

$$\lim_{p \rightarrow 0} \|f\|_{L^p} = \begin{cases} \exp \int_X \log |f| d\mu, & \text{if } \int_X \log |f| d\mu > -\infty, \\ 0 & \text{otherwise.} \end{cases}$$

HW8.3 Give an example where $0 < r < s \leq \infty$ and $L^r \subset L^s$ (this is the opposite inclusion from HW8.2b). Give conditions on a measure μ which implies one of these two inclusions always holds.

HW8.4 Suppose $1 < p < \infty$ and $f \in L^p((0, \infty))$ with respect to Lebesgue measure. For $0 < x < \infty$, set

$$F(x) = \frac{1}{x} \int_0^x f(t) dt.$$

a) Prove Hardy's inequality:

$$\|F\|_{L^p} \leq \frac{p}{p-1} \|f\|_{L^p}.$$

b) Prove equality holds if and only if $f = 0$ almost everywhere.

c) Prove that the constant $p/(p-1)$ is optimal (in other words, prove that no smaller constant can be used).

d) If $f > 0$ and $f \in L^1$ prove that $F \notin L^1$.

(Hints: (a) Assume first that $f \geq 0$, $f \in C_c$, and show you can integrate by parts to get

$$\int_0^\infty F^p dx = -p \int_0^\infty F^{p-1} x F' dx.$$

Use that in this case $x F' = f - F$, and apply Hölder's inequality to $\int F^{p-1} f dx$. Use a density argument (or some other argument) to derive the general case from this. (c) Try $f(x) = x^{-1/p}$ for $x \in [1, A]$, $f(x) = 0$ on $[1, A]^c$ and fiddle with A .