

HOMEWORK 7 FOR 18.125, SPRING 2010
DUE THURSDAY, APRIL 15 AT THE BEGINNING OF
LECTURE.
UPDATED 10 APRIL.

Note: Problem 7.1 has been updated to say $V \supset K$ instead of $V \subset K$.

HW7.1 **Definition:** Let X be a locally compact Hausdorff space. A measure μ defined on the Borel σ -algebra is called a *Borel measure*. A Borel set is *outer regular* if it satisfies (c) from the Riesz representation theorem, and *inner regular* if it satisfies (d) from the Riesz representation theorem. If every Borel set is both outer and inner regular, then μ is a *regular Borel measure*. That is, all Borel sets can be approximated from above by open sets and from below by compact sets.

Now let X be a compact Hausdorff space, and let μ be a regular Borel measure on X such that $\mu(X) = 1$. Prove there is a compact set K (called the *support* of μ) such that $\mu(K) = 1$, but $\mu(H) < 1$ for every proper compact subset $H \subset K$. *Hint:* Let K be the intersection of all compact K_α with $\mu(K_\alpha) = 1$. Show that every open set $V \supset K$ also contains one of the K_α . Show that K° is the largest open set with measure 0.

HW7.2 Show that every compact subset of $\mathbb{R}$ is the support of a Borel measure.

HW7.3 **Definition:** Let X be a locally compact Hausdorff space. A function $f : X \rightarrow \mathbb{R}$ is a *Borel function* if $f^{-1}(A)$ is a Borel set in X whenever $A \subset \mathbb{R}$ is a Borel set.

Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be a Lebesgue measurable function. Prove there exist Borel functions $g(x), h(x)$ such that $g(x) = h(x)$ Lebesgue almost everywhere, and $g(x) \leq f(x) \leq h(x)$ for every $x \in \mathbb{R}^N$. *Hint:* Recall that the σ -algebra of Lebesgue measurable sets contains the Borel algebra. How do you approximate a measurable set by Borel sets in $\mathbb{R}^N$?

HW7.4 If X is a compact topological space and $f : X \rightarrow \mathbb{R}$ is upper semi-continuous, prove that f attains a maximum on X .