

HOMEWORK 6 FOR 18.125, SPRING 2010
DUE THURSDAY, APRIL 8 AT THE BEGINNING OF
LECTURE.
UPDATED 4/7/2010.

Note: There are some typos corrected in problems 1 and 4.

HW6.1 Let X be a metric space with metric ρ . For any non-empty subset $E \subset X$, define

$$\rho_E(x) = \inf\{\rho(x, y) : y \in E\}.$$

Show that ρ_E is a uniformly continuous function on X .

If $A, B \subset X$ are disjoint **closed** subsets, define

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}.$$

Show f is continuous and describe $\text{supp } f$ and $\{f = 1\}$. Use this to provide a constructive proof of Urysohn's Lemma in this case. (**That is, if $K \subset V \subset X$ are compact and open respectively, then $\exists K \prec f \prec V$.**)

HW6.2 Let $\{f_n\}$ be a sequence of real-valued, non-negative functions on $\mathbb{R}$. Prove or disprove each of the following statements:

(a) If f_1 and f_2 are upper semicontinuous, then $f_1 + f_2$ is upper semicontinuous.

(b) If f_1 and f_2 are lower semicontinuous, then $f_1 + f_2$ is lower semicontinuous.

(c) If each f_n is upper semicontinuous, then $\sum_1^\infty f_n$ is upper semicontinuous.

(d) If each f_n is lower semicontinuous, then $\sum_1^\infty f_n$ is lower semicontinuous.

HW6.3 Let X be a metric space with metric d .

(a) Let $\{g_n\}$ be a sequence of continuous functions on X such that $0 \leq g_1 \leq g_2 \leq g_3 \leq \dots$, and define $f : X \rightarrow [0, \infty]$ by

$$f(x) = \lim_{n \rightarrow \infty} g_n(x).$$

Show that f is lower semicontinuous.

(b) Conversely, suppose $f : X \rightarrow [0, \infty]$ is lower semicontinuous and $f(p) < \infty$ for at least one $p \in X$. For $n \geq 1$ define

$$g_n(x) = \inf\{f(q) + nd(x, q) : q \in X\}.$$

Show that for each n g_n is continuous, and that $g_n \nearrow f$. (This shows that a non-negative lower semicontinuous function is the increasing limit of continuous functions.)

HW6.4 Let $X = \mathbb{R}^2$ equipped with the metric

$$\rho((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2, \\ 1 + |y_1 - y_2| & \text{if } x_1 \neq x_2. \end{cases}$$

Show that ρ is a metric and that (X, ρ) is a locally compact Hausdorff space.

If $f \in C_c(X)$, let

$$A_f = \{x \in \mathbb{R} : f(x, y) \neq 0 \text{ for at least one } y\}.$$

Show A_f is finite, and define

$$\Lambda f = \sum_{x \in A_f} \int_{-\infty}^{\infty} f(x, y) dy.$$

Let μ be the measure constructed for Λ in the Riesz Representation Theorem. If E is the x -axis, show that $\mu(E) = \infty$, but for every compact $K \subset E$, $\mu(K) = 0$. (*Hint:* First determine the compact sets in X .)