

HOMEWORK 3 FOR 18.125, SPRING 2010
DUE THURSDAY, FEBRUARY 25 AT THE BEGINNING OF
LECTURE.

HW3.1 a) Let $E = \mathbb{R}$, let

$$\tau_0 = \{\text{half-open intervals } [a, b) : a < b\},$$

and let

$$\tau = \{\text{unions of sets in } \tau_0\}.$$

Show that τ is a topology on E .

b) Show that $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ set.

c) This problem shows that G_δ is not closed under countable unions.

Step 1: Show that if $\{U_n\} \subset \mathbb{R}$ and each U_n is open and dense, then

$$\bigcap_{n=1}^{\infty} U_n$$

is dense. (This is a baby version of the Baire Category Theorem).

Step 2: Show $\mathbb{Q}$ is not a G_δ set. (*Hint:* Assume the contrary, and use part (b) plus Step 1 to conclude $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})$ is dense - a contradiction.)

Step 3: Find a sequence of $\{B_n\}_{n=1}^{\infty} \subset G_\delta$ so that

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} B_n.$$

Conclude G_δ is not closed under countable unions.

HW3.2 a) Let $\mathcal{B} = \mathcal{P}(\mathbb{R}^N)$ be the set of all sets in $\mathbb{R}^N$. Show $\mathcal{B}$ is a σ -algebra.

Let $\mu_1 : \mathcal{B} \rightarrow [0, \infty]$ be defined by

$$\mu_1(A) = \begin{cases} \infty & \text{if } A \text{ is infinite,} \\ m & \text{if } A \text{ has } m \text{ elements in it.} \end{cases}$$

Show μ_1 is a measure.

Let $\mu_2 : \mathcal{B} \rightarrow \{0, 1\}$ be defined by

$$\mu_2(A) = \begin{cases} 1 & \text{if } 0 \in A, \\ 0 & \text{if } 0 \notin A. \end{cases}$$

Show μ_2 is a measure.

b) Let $\mathcal{B}$ be the collection of sets $E \subset \mathbb{R}^N$ such that either E or E^c is at most countable. Show $\mathcal{B}$ is a σ -algebra.

Let $\mu : \mathcal{B} \rightarrow \{0, 1\}$ be defined by $\mu(E) = 0$ if E is at most countable, and $\mu(E) = 1$ if E^c is at most countable. Show μ is a measure.

HW3.3 Prove or disprove: There exists a σ -algebra $\mathcal{B}$ which is countably infinite.

HW3.4 Stroock 3.1.9

HW3.5 Stroock 3.1.10