

HOMEWORK 10 (LAST ONE!!) FOR 18.125, SPRING 2010
DUE THURSDAY, MAY 6 AT THE BEGINNING OF LECTURE.

HW10.1 **Definition:** A Hilbert space H is *separable* if it has a countable dense subset. Most of the examples which come up in “real life” are separable (for example, $L^2(\mathbb{R}^n)$).

This exercise shows that the Gram-Schmidt procedure from linear algebra can be used to construct orthonormal bases in separable Hilbert spaces.

Let H be any Hilbert space, and let $\{x_n\}_{n=1}^\infty \subset H$ be a linearly independent set of vectors. Show the following construction leads to an orthonormal set $\{u_n\}_{n=1}^\infty$ such that for each N ,

$$\text{span}\{x_1, \dots, x_N\} = \text{span}\{u_1, \dots, u_N\}.$$

Set $u_1 = x_1/\|x_1\|$. Having chosen $u_1, \dots, u_{n-1}$, let

$$v_n = x_n - \sum_{i=1}^{n-1} (x_n, u_i) u_i, \text{ and set } u_n = v_n/\|v_n\|.$$

Notice for this step, H need not be separable, but the collection of x_n s is necessarily countable.

Show any separable Hilbert space has a countable orthonormal basis.

10.2 Hilbert spaces are supposed to mimic finite dimensional Euclidean space as best they can. This exercise shows that an infinite dimensional Hilbert space does not have the Heine-Borel property.

Let H be a Hilbert space, and let $S = \{u_n\}_{n=1}^\infty$ be an orthonormal set in H .

a) Show that S is closed and bounded but not compact. Show this implies the unit ball is not compact. Show this implies H is not locally compact.

b) Let $\{\delta_n\}_{n=1}^\infty$ be a sequence of positive real numbers, and let S_1 be the set of all $x \in H$ such that

$$x = \sum_{n=1}^\infty c_n u_n, \text{ with } |c_n| \leq \delta_n.$$

Show that S_1 is compact if and only if $\sum_{n=1}^\infty \delta_n^2 < \infty$.

HW10.3 In class we showed a closed convex subset of a Hilbert space has a unique element of smallest norm. The analogous statement is false in Banach spaces, as this exercise shows.

Let $M \subset L^1([0, 1])$ (with Lebesgue measure) be the subset of functions $f \in L^1([0, 1])$ such that

$$\int_0^1 f dx = 1.$$

Show that M is closed and convex, but contains infinitely many elements of minimum norm.

HW10.4 This exercise proves the *closed graph theorem*.

Suppose X, Y are Banach spaces, and suppose $\Lambda : X \rightarrow Y$ is linear. Let $X \oplus Y$ be the set of all ordered pairs $(x, y) \in X \times Y$ with addition and scalar multiplication defined component-wise. Show $X \oplus Y$ is a Banach space with norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$.

Let $\Gamma(\Lambda)$ be the graph of Λ :

$$X \times Y \supset \Gamma(\Lambda) = \{(x, y) \in X \times Y : y = \Lambda(x)\}.$$

Show that $\Gamma(\Lambda)$ is closed if and only if Λ is continuous.

Hint: One direction is straightforward. For the other direction, if $\Gamma(\Lambda)$ is closed, show it is a Banach space. The map $(x, \Lambda x) \mapsto x$ is continuous and bijective as a mapping $\Gamma(\Lambda) \rightarrow X$. Apply a theorem from lecture (on 4 May).