Fifth homework assignment in 18.101

(1) Let $V_k(\mathbb{R}^n)$ be the set of all k-tuples of vectors, $\mathrm{v}_1, \ldots, \mathrm{v}_k$ in $\mathbb{R}^n$ which are mutually orthonormal, i.e., satisfy $\langle \mathrm{v}_i, \mathrm{v}_j \rangle = \delta_{i,j}$ for $1 \leq i, j \leq k$. Show that $V_k(\mathbb{R}^n)$ is a d-dimensional manifold where $d = k\left(n - \frac{k+1}{2}\right)$.

Hints:

(a) Let $M_{k,n}$ be the set of all $k \times n$ matrices. Show that $V_k(\mathbb{R}^n)$ can be identified with the set

$$\{A \in M_{k,n} \qquad AA^t = I_k\}$$

where I_k is the identity $k \times k$ matrix.

(b) Let S_k be the set of symmetric $k \times k$ matrices and let $\phi : M_{k,n} \to S_k$ be the map, $\phi(A) = AA^t$.

Show that I_k is a regular value of this map.

Subhint: Show that $d\phi_A$ can be identified with the map

$$M_{k,n} \to S_k\,, \quad B \to BA^t + AB^t\,.$$

(2) Lecture 7, problem 3. Lecture 8, problem 5.

(3) Lecture 9, problems 4 and 5.

(4) Lecture 9, problem 10.