The theory of manifolds Lecture 3

Definition 1. *The tangent space of an open set $U \subset \mathbb{R}^n$, TU is the set of pairs $(x, v) \in U \times \mathbb{R}^n$. This should be thought of as a vector v based at the point $x \in U$. Denote by $T_pU \subset TU$ the vector space consisting of all vectors (p, v) based at the point p. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ the tangent map of f is defined by*

$$Tf : T\mathbb{R}^n \longrightarrow T\mathbb{R}^m$$

$$Tf(x, v) := (f(x), Df(x)v)$$

We also define the linear map

$$T_pf : T_p\mathbb{R}^n \longrightarrow T_{f(p)}\mathbb{R}^m$$

$$T_pf(p, v) := (f(p), Df(p)v)$$

Recall that the chain theorem tells us that

$$T(f \circ g) = Tf \circ Tg$$

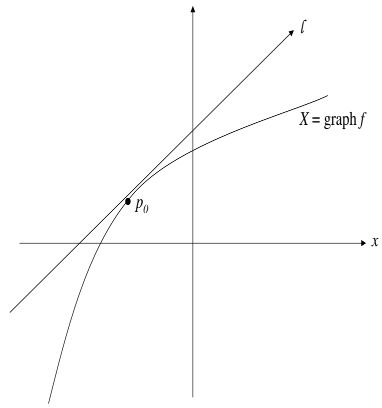
We recall that a subset, X, of $\mathbb{R}^N$ is an n-dimensional manifold, if, for every $p \in X$, there exists an open set, $U \subseteq \mathbb{R}^n$, a neighborhood, V, of p in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$-diffeomorphism, $\varphi : U \to X \cap X$.

Definition 2. *We will call φ a* parameterization *of X at p.*

Our goal in this lecture is to define the notion of the *tangent space*, T_pX, to X at p and describe some of its properties. Before giving our official definition we'll discuss some simple examples.

Example 1.

Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function and let $X = \text{graph} f$.



Then in this figure above the tangent line, ℓ, to X at $p_0 = (x_0, y_0)$ is defined by the equation

$$y - y_0 = a(x - x_0)$$

where $a = f'(x_0)$ In other words if p is a point on ℓ then $p = p_0 + \lambda \mathrm{v}_0$ where $\mathrm{v}_0 = (1, a)$ and $\lambda \in \mathbb{R}^n$. We would, however, like the tangent space to X at p_0 to be a subspace of the tangent space to $\mathbb{R}^2$ at p_0, i.e., to be the subspace of the space: $T_{p_0}\mathbb{R}^2 = \{p_0\} \times \mathbb{R}^2$, and this we'll achieve by defining

$$T_{p_0}X = \{(p_0, \lambda \mathrm{v}_0)\,, \quad \lambda \in \mathbb{R}\}\,.$$

Example 2.

Let S^2 be the unit 2-sphere in $\mathbb{R}^3$. The tangent plane to S^2 at p_0 is usually defined to be the plane

$$\{p_0 + \mathrm{v}\,;\, v \in \mathbb{R}^3\,, \quad \mathrm{v} \perp p_0\}\,.$$

However, this tangent plane is easily converted into a subspace of $T_p\mathbb{R}^3$ via the map, $p_0 + \mathrm{v} \to (p_0, \mathrm{v})$ and the image of this map

$$\{(p_0, \mathrm{v})\,;\, \mathrm{v} \in \mathbb{R}^3\,, \quad \mathrm{v} \perp p_0\}$$

will be our definition of $T_{p_0}S^2$.

Let's now turn to the general definition. As above let X be an n-dimensional submanifold of $\mathbb{R}^N$, p a point of X, V a neighborhood of p in $\mathbb{R}^N$, U an open set in $\mathbb{R}^n$ and

$$\varphi : (U, q) \to (X \cap V, p)$$

a parameterization of X. We can think of φ as a $\mathcal{C}^\infty$ map

$$\varphi : (U, q) \to (V, p)$$

whose image happens to lie in $X \cap V$ and we proved last time that its derivative at q

$$T_q\varphi : T_q\mathbb{R}^n \to T_p\mathbb{R}^N \tag{1}$$

is injective.

Definition 3. *The tangent space, T_pX, to X at p is the image of the linear map (1). In other words, $w \in T_p\mathbb{R}^N$ is in T_pX if and only if $w = T_q\varphi(\mathrm{v})$ for some $\mathrm{v} \in T_q\mathbb{R}^n$. More succinctly,*

$$T_pX = T\varphi(T_q\mathbb{R}^n)\,. \tag{2}$$

(Since $T_q\varphi$ is injective this space is an n-dimensional vector subspace of $T_p\mathbb{R}^N$.)

One problem with this definition is that it appears to depend on the choice of φ. To get around this problem, we'll give an alternative definition of T_pX. Last time we showed that there exists a neighborhood, V, of p in $\mathbb{R}^N$ (which we can without loss of generality take to be the same as V above) and a $\mathcal{C}^\infty$ map

$$f : (V, p) \to (\mathbb{R}^k, 0)\,, \quad k = N - n\,, \tag{3}$$

such that $X \cap V = f^{-1}(0)$ and such that f is a submersion at all points of $X \cap V$, and in particular at p. Thus

$$T_pf : T_p\mathbb{R}^N \to T_0\mathbb{R}^k$$

is surjective, and hence the kernel of T_pf has dimension n. Our alternative definition of T_pX is

$$T_pX = \text{kernel } \; T_pf\,. \tag{4}$$

The spaces (2) and (4) are both n-dimensional subspaces of $T_p\mathbb{R}^N$, and we claim that these spaces are the same. (Notice that the definition (4) of T_pX doesn't depend on φ, so if we can show that these spaces are the same, the definitions (2) and (4) will depend *neither* on φ *nor* on f.)

Proof. Since $\varphi(U)$ is contained in $X \cap V$ and $X \cap V$ is contained in $f^{-1}(0)$, $f \circ \varphi = 0$, so by the chain rule

$$T_pf \circ T_qf = T_q(f \circ \varphi) = 0\,. \tag{5}$$

Hence if $\mathrm{v} \in T_p\mathbb{R}^n$ and $w = T\varphi(\mathrm{v})$, $T_pf(w) = 0$. This shows that the space (2) is contained in the space (4). However, these two spaces are n-dimensional so the coincide.

$\square$

From the proof above one can extract a slightly stronger result:

Theorem 1. *Let W be an open subset of $\mathbb{R}^\ell$ and $h : (W, q) \to (\mathbb{R}^N, p)$ a $\mathcal{C}^\infty$ map. Suppose $h(W)$ is contained in X. Then the image of the map*

$$T_q h : T_q\mathbb{R}^\ell \to T_p\mathbb{R}^N$$

is contained in T_pX.

Proof. Let f be the map (3). We can assume without loss of generality that $h(W)$ is contained in V, and so, by assumption, $h(W) \subseteq X \cap V$. Therefore, as above, $f \circ h = 0$, and hence $T_q h(T_q\mathbb{R}^\ell)$ is contained in the kernel of T_pf.

□

Definition 4. *Define the tangent space to a manifold $X \subset \mathbb{R}^N$, to be the subset $TX \subset T\mathbb{R}^N$ given by*

$$\{(x, v) \subset T\mathbb{R}^N \text{ so that } (x, v) \in T_xX \text{ for some } x \in X\}$$

Theorem 2. *If $X \subset \mathbb{R}^N$ is a smooth sub manifold of $\mathbb{R}^N$, then $TX \subset T\mathbb{R}^N$ is a smooth sub manifold.*

The proof of this is left as an exercise.

We shall now define the tangent map or *derivative* of a mapping between manifolds. Explicitly: Let X be a submanifold of $\mathbb{R}^N$, Y a submanifold of $\mathbb{R}^m$ and $g : (X, p) \to (Y, y_0)$ a $\mathcal{C}^\infty$ map. There exists a neighborhood, $\mathcal{O}$, of X in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$ map, $\tilde{g} : \mathcal{O} \to \mathbb{R}^m$ extending to g. We will define

$$Tg : TX \longrightarrow TY$$

to be the restriction of the map

$$T\tilde{g}$$

to $T(X \cap \mathcal{O}) \subset T\mathcal{O}$

$$(T_pg) : T_pX \to T_{y_0}Y \tag{6}$$

to be the restriction of the map

$$T_p\tilde{g} : T_p\mathbb{R}^N \to T_{y_0}\mathbb{R}^m \tag{7}$$

to T_pX. There are two obvious problems with this definition:

1. Is the space
$$T_p\tilde{g}(T_pX)$$
contained in $T_{y_0}Y$?

2. Does the definition depend on $\tilde{g}$?

To show that the answer to 1. is yes and the answer to 2. is no, let

$$\varphi : (U, x_0) \to (X \cap V, p)$$

be a parameterization of X, and let $h = \tilde{g} \circ \varphi$. Since $\varphi(U) \subseteq X$, $h(U) \subseteq Y$ and hence by Theorem 2

$$T_{x_0} h(T_{x_0}\mathbb{R}^n) \subseteq T_{y_0} Y \, .$$

But by the chain rule

$$T_{x_0} h = T_p \tilde{g} \circ T_{x_0} \varphi \, , \tag{8}$$

so by (2)

$$(T_p\tilde{g})(T_pX) \quad \subseteq \quad T_pY \tag{9}$$

and

$$(T_p\tilde{g})(T_pX) \quad = \quad T_{x_0} h(T_{x_0}\mathbb{R}^n) \tag{10}$$

Thus the answer to 1. is yes, and since $h = \tilde{g} \circ \varphi = g \circ \varphi$, the answer to 2. is no.

From (5) and (6) one easily deduces

Theorem 3 (Chain rule for mappings between manifolds)**.** *Let Z be a submanifold of $\mathbb{R}^\ell$ and $\psi : (Y, y_0) \to (Z, z_0)$ a $\mathcal{C}^\infty$ map. Then $T_q\psi \circ T_pg = T_p(\psi \circ g)$.*

Problem set

1. What is the tangent space to the quadric, $x_n^2 = x_1^2 + \cdots + x_{n-1}^2$, at the point, $(1, 0, \ldots, 0, 1)$?

2. Show that the tangent space to the $(n-1)$-sphere, S^{n-1}, at p, is the space of vectors, $(p, \mathrm{v}) \in T_p\mathbb{R}^n$ satisfying $p \cdot \mathrm{v} = 0$.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map and let $X = \mathrm{graph} f$. What is the tangent space to X at $(a, f(a))$?

4. Let $\sigma : S^{n-1} \to S^{n-1}$ be the anti-podal map, $\sigma(x) = -x$. What is the derivative of σ at $p \in S^{n-1}$?

5. Let $X_i \subseteq \mathbb{R}^{N_i}$, $i = 1, 2$, be an n_i-dimensional manifold and let $p_i \in X_i$. Define X to be the Cartesian product

$$X_1 \times X_2 \subseteq \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

and let $p = (p_1, p_2)$. Describe T_pX in terms of $T_{p_1}X_1$ and $T_{p_2}X_2$.

6. Let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold and $\varphi_i : U_i \to X \cap V_i$, $i = 1, 2$. From these two parameterizations one gets an overlap diagram

$$\begin{tikzcd} & X \cap V & \\ W_1 \arrow[ur, "\varphi_1"] \arrow[rr, "\psi"] & & W_2 \arrow[ul, "\varphi_2"'] \end{tikzcd} \tag{11}$$

where $V = V_1 \cap V_2$, $W_i = \varphi_i^{-1}(X \cap V)$ and $\psi = \varphi_2^{-1} \circ \varphi_1$.

(a) Let $p \in X \cap V$ and let $q_i = \varphi_i^{-1}(p)$. Derive from the overlap diagram (10) an overlap diagram of linear maps

$$\begin{tikzcd} & T_p\mathbb{R}^N \arrow[dr, "(T\varphi_2)"] & \\ T_{q_1}\mathbb{R}^n \arrow[ur, "(T\varphi_1)"] \arrow[rr, "(T\psi)"] & & T_{q_2}\mathbb{R}^n \end{tikzcd} \tag{12}$$

(b) Use overlap diagrams to give another proof that T_pX is intrinsically defined.