# Problem Set 9, 18.100B/C, Fall 2011

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## 1

### a

Recall from problem set 8 that if $f : [0, \infty) \longrightarrow [0, \infty)$ is a continuous, strictly increasing function and $f(0) = 0$ then

$$\int_0^u f(t)dt + \int_0^v f^{-1}(t)dt \geq uv$$

with equality if and only if $f(u) = v$.

Let $p, q$ be strictly positive real numbers such that $1/p + 1/q = 1$. Multiplying by $pq$ gives

$$p + q = pq \implies (p-1)(q-1) = pq - p - q + 1 = 1.$$

Thus defining $f : [0, \infty) \longrightarrow [0, \infty)$ by $f(t) = t^{p-1}$, $f^{-1} : [0, \infty) \longrightarrow [0, \infty)$ is given by $f(t) = t^{q-1}$ and since $f$ is continuous and strictly increasing we obtain

$$\int_0^u t^{p-1}dt + \int_0^v t^{q-1}dt \geq uv.$$

with equality if and only if $u^{p-1} = v$. Noting that

$$u^{p-1} = v \iff u^p = u^{q(p-1)} = v^q$$

and using the Fundamental Theorem of Calculus, we're done.

### b

Let $f, g \in \mathscr{R}(\alpha)$ with $f, g \geq 0$. Then applying part $(a)$ pointwise we obtain

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$

$f^p, g^q \in \mathscr{R}(\alpha)$ by theorem 6.11 and theorem 6.12$a)b)$ gives

$$\int_a^b fg d\alpha \leq \frac{1}{p}\int_a^b f^p d\alpha + \frac{1}{q}\int_a^b g^q d\alpha.$$

**c**

For $f, g \in \mathscr{R}(\alpha)$, $|f|^p, |g|^q \in \mathscr{R}(\alpha)$ by theorems 6.13$b$) and 6.11. Let $\|f\|_{L_p} = \left(\int_a^b |f|^p d\alpha\right)^{\frac{1}{p}}$. If $\|f\|_{L_p}$ and $\|g\|_{L_q}$ are non-zero then

$$\int_a^b \left(|f|/\|f\|_{L_p}\right)^p d\alpha = 1 = \int_a^b \left(|g|/\|g\|_{L_q}\right)^q d\alpha$$

So by ($b$),

$$\int_a^b \left|\frac{fg}{\|f\|_{L_p}\|g\|_{L_q}}\right| \leq 1$$

giving

$$\left|\int_a^b fg d\alpha\right| \leq \int_a^b |fg| d\alpha \leq \|f\|_{L_p}\|g\|_{L_q}.$$

Suppose $\|f\|_{L_p} = 0$ and let $c > 0$. Part ($a$) gives

$$c|fg| \leq \frac{|f|^p}{p} + \frac{c^q|g|^q}{q}$$

and thus

$$\left|\int_a^b fg d\alpha\right| \leq \int_a^b |fg| d\alpha \leq c^{q-1}\int_a^b \frac{|g|^q}{q} d\alpha.$$

Letting $c \longrightarrow 0$ gives $\left|\int_a^b fg d\alpha\right| = 0$ because $q > 1$. Thus the inequality remains valid. Similarly, it remains valid if $\|g\|_{L_q} = 0$.

## 2

Let $x > 0$, let $n \in \mathbb{N} \cup \{0\}$, and let $f : [0, x] \longrightarrow \mathbb{R}$ be $n+1$ times differentiable with $f^{(n+1)}$ integrable.

First note that for $m \leq n$, $f^{(m)}$ is differentiable, hence continuous, hence integrable. Thus $f^{(m)}$ is integrable whenever $m \leq n+1$ and so for $m \leq n$

$$I_m(x) = \frac{x^{m+1}}{m!}\int_0^1 (1-t)^m f^{(m+1)}(tx) dt = \frac{1}{m!}\int_0^x (x-y)^m f^{(m+1)}(y) dy$$

makes sense, the equality between the integrals coming from making the substitution $y = tx$.

By the fundamental theorem of calculus we have

$$I_0(x) = \int_0^x f'(y) dy = f(x) - f(0) \implies f(x) = f(0) + I_0(x).$$

Suppose inductively that for $0 < m \leq n$ we have

$$f(x) = f(0) + f'(0) + f''(0)x + \ldots + \frac{f^{(m-1)}(0)}{(m-1)!}x^{m-1} + I_{m-1}(x).$$

We would like to show that

$$f(x) = f(0) + f'(0) + f''(0)x + \ldots + \frac{f^{(m)}(0)}{m!}x^m + I_m(x)$$

and for this it is enough to show

$$I_{m-1}(x) = \frac{f^{(m)}(0)}{m!}x^m + I_m(x).$$

This follows from theorem 6.22 with $F(y) = \frac{(x-y)^m}{m!}$ and $G(y) = f^{(m)}(y)$.

## 3

Let $F(x) = xf(x)$ and $G(x) = f(x)$. Then theorem 6.22 (everything in sight is bounded and continuous and hence integrable by theorem 6.8) gives

$$\int_a^b xf(x)f'(x)dx = -\int_a^b (f(x) + xf'(x))f(x)dx = -\int_a^b f(x)^2dx - \int_a^b xf(x)f'(x)dx$$

$$\implies \int_a^b xf(x)f'(x)dx = -\frac{1}{2}\int_a^b f(x)^2dx = -\frac{1}{2}$$

Applying $1)c)$ with $p = q = 2$ we obtain

$$\frac{1}{4} = \left(\int_a^b xf(x)f'(x)dx\right)^2 \leq \int_a^b f'(x)^2dx \int_a^b x^2f(x)^2dx.$$

If equality holds then $f'(x) = \lambda xf(x)$ for some $\lambda$. Let $g(x) = f(x)e^{-\lambda x^2/2}$. Thus

$$g'(x) = f'(x)e^{-\lambda x^2/2} - \lambda xf(x)e^{-\lambda x^2/2} = 0.$$

By theorem $5.11)b)$, $g$ is constant. Thus $f(x) = Ce^{-\lambda x^2/2}$ for some constant $C$. Since $f(a) = f(b) = 0$, $C = 0$ and thus $f = 0$. But

$$\int_a^b f(x)^2dx = 1.$$

We have a contradiction and so the inequality is strict.

## 4

Suppose $(f_n)$ and $(g_n)$ converge uniformly on some set $E$. Let their respective limits be $f$ and $g$ and let $\epsilon > 0$. There exist $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |f_n(x) - f(x)| \leq \epsilon/2 \text{ for all } x \in E$$

and

$$n \geq N_2 \implies |g_n(x) - g(x)| \leq \epsilon/2 \text{ for all } x \in E.$$

Let $N = \max\{N_1, N_2\}$. Then

$$n \geq N \implies |(f_n + g_n)(x) - (f+g)x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \leq \epsilon, \text{ for all } x \in E.$$

Thus $(f_n + g_n)$ converges uniformly on $E$.

Suppose in addition that $(f_n)$ and $(g_n)$ are sequences of bounded functions. Then there exists $M_1, M_2 \in \mathbb{N}$ such that $|f_{N_1}(x)| \leq M_1 - \epsilon/2$ and $|g_{N_1}(x)| \leq M_2 - \epsilon/2$ for all $x \in E$. Thus

$$|f(x)| \leq |f(x) - f_{N_1}(x)| + |f_{N_1}(x)| \leq M_1$$

and

$$|g(x)| \leq |g(x) - g_{N_2}(x)| + |g_{N_2}(x)| \leq M_2$$

for all $x \in E$. Thus, for any $n \in \mathbb{N}$ and $x \in E$,

$$\begin{aligned} |(f_n g_n)(x) - (fg)(x)| &\leq |f_n(x)||g_n(x) - g(x)| + |f_n(x) - f(x)||g(x)| \\ &\leq |f_n(x) - f(x)||g_n(x) - g(x)| + |f(x)||g_n(x) - g(x)| + |f_n(x) - f(x)||g(x)| \\ &\leq |f_n(x) - f(x)||g_n(x) - g(x)| + M_1|g_n(x) - g(x)| + M_2|f_n(x) - f(x)| \end{aligned}$$

Given $\delta > 0$, choose $\tilde{N}_1, \tilde{N}_2 \in \mathbb{N}$ such that

$$n \geq \tilde{N}_1 \implies |f_n(x) - f(x)| \leq \min\{\delta, 1\}/(3M_2) \quad \text{for all } x \in E$$

and

$$n \geq \tilde{N}_2 \implies |g_n(x) - g(x)| \leq \delta/(3M_1) \quad \text{for all } x \in E.$$

Let $\tilde{N} = \max\{\tilde{N}_1, \tilde{N}_2\}$. Then

$$n \geq \tilde{N} \implies |(f_n g_n)(x) - (fg)(x)| \leq \delta \quad \text{for all } x \in E.$$

Thus $(f_n g_n)$ converges uniformly on $E$.

# 5

For each $n \in \mathbb{N}$ define $f_n : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x + 1/n$ and define $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x$.

Let $\epsilon > 0$ and choose $N > 1/\epsilon$. Then $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in \mathbb{R}$ so that $(f_n)$ converges uniformly to $f$ on $\mathbb{R}$.

Let $g_n = f_n$ and $g = f$. Pointwise $f_n g_n \longrightarrow fg$ so that if $(f_n g_n)$ were to convergence uniformly there would exist an $N \in \mathbb{N}$ such that

$$n \geq N \implies |(f_n g_n)(x) - (fg)(x)| \leq 1 \quad \text{for all } x \in \mathbb{R}.$$

However,

$$(f_n g_n)(x) - (fg)(x) = (x + 1/n)^2 - x^2 = 2x/n + 1/n^2$$

so that

$$|(f_n g_n)(n) - (fg)(n)| \geq 2.$$

# 6

Let $f : [0,1] \longrightarrow [0,1]$ be continuously differentiable with nonincreasing derivative and with $f(0) = f(1) = 0$. The graph of $f$ gives an arc in $[0,1]^2$:

$$\gamma : [0,1] \longrightarrow [0,1]^2, \ \ t \longmapsto (t, f(t))$$

$\gamma$ is continuously differentiable with

$$\gamma' : [0,1] \longrightarrow \mathbb{R}^2, \ \ t \longmapsto (1, f'(t)).$$

By theorem 6.27 the arc length of $\gamma$ is given by

$$\int_0^1 |\gamma'(t)| dt = \int_0^1 \sqrt{1 + f'(t)^2} dt.$$

The Cauchy-Schwartz inequality proved on an earlier sheet gives $\sqrt{1 + f'(t)^2} \leq 1 + |f'(t)|$. So that the length of $\gamma$ is bounded by

$$\int_0^1 1 + |f'(t)| dt.$$

If $f'(t) > 0$ for all $t \in [0,1]$ question 4 from sheet 8 and the Fundamental Theorem of Calculus give

$$f(1) - f(0) = \int_0^1 f'(t) dt > 0,$$

a contradiction. So $\{t \in [0,1] : f'(t) \leq 0\} \neq \emptyset$. Similarly, $\{t \in [0,1] : f'(t) \geq 0\} \neq \emptyset$. Since $f'$ is continuous the intermediate value theorem gives an $a \in [0,1]$ such that $f'(a) = 0$. Since $f'$ is non-increasing $f'(x) \geq 0$ on $[0,a]$ and $f'(x) \leq 0$ on $[a,1]$. Thus the fundamental theorem of calculus gives

$$\int_0^1 1 + |f'(t)| dt = 1 + \int_0^a f'(t) dt - \int_a^1 f'(t) dt = 1 + (f(a) - f(0)) - (f(1) - f(a)) = 1 + 2f(a) \leq 3.$$