Problem Set 8, 18.100B/C, Fall 2011

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1.1

Suppose $a \in \mathbb{R}$ and that $f : (a, \infty) \longrightarrow \mathbb{R}$ is a twice-differentiable. Suppose, in addition that

$$\sup_{x \in (a,\infty)} |f(x)| = M_0, \quad \sup_{x \in (a,\infty)} |f'(x)| = M_1 \quad \text{and} \quad \sup_{x \in (a,\infty)} |f''(x)| = M_2.$$

Let $x \in (a, \infty)$ and $h > 0$. Taylor's Theorem (5.15) gives us a $\xi \in (x, x + 2h)$ such that

$$f(x + 2h) = f(x) + 2hf'(x) + (2h)^2 \frac{f''(\xi)}{2}$$

and so

$$2h|f'(x)| = |f(x + 2h) - f(x) - 2h^2 f''(\xi)| \leq 2M_0 + 2h^2 M_2.$$

Hence, for all $h > 0$, $hM_1 \leq M_0 + h^2 M_2$. When $h \leq 0$, $hM_1 \leq 0 \leq M_0 + h^2 M_2$ and

$$h^2 M_2 - hM_1 + M_0 \geq 0 \quad \text{for all} \quad h \in \mathbb{R}.$$

This means the discriminant is less than or equal to 0, i.e. $M_1^2 \leq 4M_0 M_2$.

1.2

Define $f : (-1, \infty) \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 2x^2 - 1 & x \in (-1, 0] \\ \frac{x^2-1}{x^2+1} & x \in [0, \infty) \end{cases}$$

Then

$$f'(x) = \begin{cases} 4x & x \in (-1, 0] \\ \frac{4x}{(x^2+1)^2} & x \in [0, \infty) \end{cases}$$

and

$$f''(x) = \begin{cases} 4 & x \in (-1, 0] \\ \frac{4(1-3x^2)}{(x^2+1)^3} & x \in [0, \infty) \end{cases}$$

1. $M_0 = 1$. $\left|\frac{x^2-1}{x^2+1}\right| \le \frac{x^2+1}{x^2+1} = 1$ and $|2x^2 - 1| \le 1$ on $(-1, 0]$. Also $|f(0)| = 1$.
2. $M_1 = 4$. $|x| \le (x^2+1)^2$ is seen by considering the cases $|x| \le 1$ and $|x| \ge 1$, which gives $\left|\frac{4x}{(x^2+1)^2}\right| \le 4$. $\sup_{x\in(-1,0]} |4x| = 4$.
3. $M_2 = 4$. We just need to show that $|1-3x^2| \le |x^2+1|^2$. This is true if and only if $(1-3x^2)^2 \le (x^2+1)^4$ and this follows from $x^2(x^2+5)((x^2-1/2)^2 + 7/8) \ge 0$.

1.3

For vector-valued functions... I have a feeling this is false, but cannot find a counter example.

2

Suppose $f : (0, \infty) \longrightarrow \mathbb{R}$ is twice differentiable, that f'' is bounded by M and that $f(x) \longrightarrow 0$ as $x \longrightarrow \infty$.

Let $a \in (0, \infty)$. By the first question,

$$\left(\sup_{x\in(a,\infty)} |f'(x)|\right)^2 \le 4 \sup_{x\in(a,\infty)} |f(x)| \sup_{x\in(a,\infty)} |f''(x)| \le 4M \sup_{x\in(a,\infty)} |f(x)|$$

$\sup_{x\in(a,\infty)} |f(x)| \longrightarrow 0$ as $a \longrightarrow \infty$ and so $\sup_{x\in(a,\infty)} |f'(x)| \longrightarrow 0$. This means $f'(x) \longrightarrow 0$ as $x \longrightarrow \infty$.

3

Suppose $\alpha : [a, b] \longrightarrow \mathbb{R}$ is increasing and continuous at $x_0 \in [a, b]$. Define $f : [a, b] \longrightarrow \mathbb{R}$ by $f(x_0) = 1$ and $f(x) = 0$ whenever $x \ne x_0$. f is bounded on $[a, b]$ and has one discontinuity at $x_0 \in [a, b]$, where α is continuous. Theorem 6.10 tells us $f \in \mathscr{R}(\alpha)$. By definition

$$\int_a^b f d\alpha = \sup L(P, f, \alpha)$$

but

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i.$$

Clearly, all the m_i are zero and so $\int f d\alpha = 0$.

4

Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is continuous. Then by theorem 6.8, $f \in \mathscr{R}$. Suppose $f \ge 0$ and that there exists an $x_0 \in [a, b]$ with $f(x_0) \ne 0$. Because f is continuous at x_0 there exists a $\delta > 0$ such that

$$x \in [a, b],\ |x - x_0| < \delta \implies |f(x) - f(x_0)| < |f(x_0)|/2.$$

Thus

$$x \in [a,b],\ |x - x_0| < \delta \implies |f(x)| > |f(x_0)|/2.$$

By definition

$$\int_a^b f(x)dx = \sup L(P,f) \quad \text{and} \quad L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i.$$

If we choose a partition P with two points in $[a,b] \cap (x_0 - \delta, x_0 + \delta)$ then at least one of the m_i's is strictly greater than zero and the rest are all greater than or equal to zero. Thus for this partition $L(P,f) > 0$ and we obtain $\int f > 0$.

By the contrapositive we have answered the question.

5

Define $f : [0,1] \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ -1 & x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q}) \end{cases}$$

Then $f(x)^2 = 1$ for all $x \in [0,1]$ and thus by Theorem 6.8, $f^2 \in \mathscr{R}$. However, $f \notin \mathscr{R}$ since one can easily check that

$$U(P,f) = 1 \quad \text{and} \quad L(P,f) = -1 \quad \text{for all partitions } P.$$

Define $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^{1/3}$. Then φ is continuous. Suppose that $f^3 \in \mathscr{R}$. Then by Theorem 6.11, $f = \varphi \circ f^3 \in \mathscr{R}$.

6

Let P be the Cantor set and let $f : [0,1] \longrightarrow \mathbb{R}$ be a bounded function, which is continuous at every point outside P. We will show that $f \in \mathscr{R}$. Reading the proof of Theorem 6.10 we see that that it will carry through provided that for any $\epsilon > 0$ we can can find finitely many disjoint intervals $[u_j, v_j] \subset [0,1]$, such that $\sum_{j=1}^m (v_j - u_j) < \epsilon$, $P \subset \bigcup_{j=1}^m [u_j, v_j]$, and $P \setminus \{0,1\} \subset \bigcup_{j=1}^m (u_j, v_j)$.

Constructing such intervals $[u_j, v_j]$ is straightforward. Given $\epsilon > 0$ we may choose $n \in \mathbb{N}$ such that $(2/3)^n < \epsilon/2$. $P \subset E_n$ (see page 41 for notation) and E_n is a union of disjoint intervals $[u_j, v_j]$ with $\sum_{j=1}^m v_j - u_j = (2/3)^n$. Replace $[0, 1/3^n]$ by $[0, 1/3^n + \delta/2m]$, $[1 - 1/3^n, 1]$ by $[1 - 1/3^n - \delta/2m, 1]$ and the other intervals $[u_j, v_j]$ by $[u_j - \delta/4m, v_j + \delta/4m]$ where $\delta < \epsilon$ is chosen so that the intervals remain disjoint.

7

Let $f : [0, \infty) \longrightarrow [0, \infty)$ be continuous and strictly increasing and suppose $f(0) = 0$. We wish to show that

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab$$

for any $a, b > 0$.

We remark that the integrals are well-defined by Theorem 6.9. By definition, there exists a sequence of partitions (P_n) of $[0, a]$ such that

$$U(P_n, f) \longrightarrow \int_0^a f(x)dx$$

and a sequence of partitions (Q_n) of $[0, b]$ such that

$$L(Q_n, f^{-1}) \longrightarrow \int_0^b f^{-1}(x)dx.$$

Let $P'_n = P_n \cap f^{-1}(Q_n)$, $\tilde{P}_n = P'_n \cap [0, a]$, $Q'_n = Q_n \cap f(P_n)$ and $\tilde{Q}_n = Q'_n \cap [0, b]$. Then $U(\tilde{P}_n, f) \leq U(P_n, f)$, $L(\tilde{Q}_n, f^{-1}) \geq L(Q_n, f^{-1})$ and so we have

$$U(\tilde{P}_n, f) \longrightarrow \int_0^a f(x)dx \quad \text{and} \quad L(\tilde{Q}_n, f^{-1}) \longrightarrow \int_0^b f^{-1}(x)dx.$$

Replace P_n by $\tilde{P}_n$ and Q_n by $\tilde{Q}_n$. Because f is strictly increasing

$$U(P_n, f) = \sum_{i=1}^{p} f(x_i)(x_i - x_{i-1})$$

where $P_n = \{x_0, \ldots, x_p\}$ and similarly,

$$L(Q_n, f^{-1}) = \sum_{j=1}^{q} f^{-1}(y_{j-1})(y_j - y_{j-1})$$

where $Q_n = \{y_0, \ldots, y_q\}$. Also,

$$ab = \left[\sum_{i=1}^{p}(x_i - x_{i-1})\right] \left[\sum_{j=1}^{q}(y_j - y_{j-1})\right]$$

Let $P'_n = \{x_0, \ldots, x_p, \ldots, x_{p'}\}$ and $Q'_n = \{y_0, \ldots, y_q, \ldots, y_{q'}\}$. Realising that we can write

$$\begin{aligned} U(P_n, f) &= \sum_{i=1}^{p} \sum_{y_j \leq f(x_i)} (y_j - y_{j-1})(x_i - x_{i-1}) \\ &= \sum_{i=1}^{p} \sum_{y_j \leq f(x_i)} (x_i - x_{i-1})(y_j - y_{j-1}) \end{aligned}$$

and

$$\begin{aligned} L(Q_n, f^{-1}) &= \sum_{j=1}^{q} \sum_{x_i \leq f^{-1}(y_{j-1})} (x_i - x_{i-1})(y_j - y_{j-1}) \\ &= \sum_{j=1}^{q} \sum_{f(x_i) < y_j} (x_i - x_{i-1})(y_j - y_{j-1}) \end{aligned}$$

we obtain $ab \leq U(P_n, f) + L(Q_n, f^{-1})$ since the left hand side involves summing over fewer terms. Taking limits gives the required identity.

When $f(a) = b$ we obtain equality because the sums are equal. To see the converse we use an argument like the one in question 4: if $f(a) > b$, find a neigborhood N on which this is true; one of the sums in the above argument will consist of more terms and the extra contribution will not converge to zero as this contribution is giving $\int_N f(x) - b dx > 0$. Simarly, if $f(a) < b$.

I would encourage drawing a picture to follow this argument. I hope it is not too hard to follow, but it might be confusing without a good picture in front of you.