Problem Set 7, 18.100B/C, Fall 2011

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1

Suppose $f : X \longrightarrow Y$ is continuous and let $E \subset X$.

$$f(E) \subset \overline{f(E)} \implies E \subset f^{-1}(\overline{f(E)})$$

By the corollary to Theorem 4.8, $f^{-1}(\overline{f(E)})$ is closed in X. Thus

$$\overline{E} \subset f^{-1}(\overline{f(E)}) \implies f(\overline{E}) \subset \overline{f(E)}.$$

Suppose $f : X \longrightarrow Y$ and that for all $E \subset X$, $f(\overline{E}) \subset \overline{f(E)}$. Let $V \subset Y$ be closed. Taking $E = f^{-1}(V)$ we see that

$$f(\overline{f^{-1}(V)}) \subset \overline{f(f^{-1}(V))} \subset \overline{V} = V.$$

Thus $\overline{f^{-1}(V)} \subset f^{-1}(V)$, which shows $f^{-1}(V)$ is closed. By the corollary to Theorem 4.8, f is continuous.

2

Let f be a continuous mapping of a compact metric space X into a metric space Y.

Suppose f is not uniformly continuous. Then there exists an $\epsilon > 0$ for which the definition of uniform continuity fails. In particular, for any $n \in \mathbb{N}$ there exist $p_n, q_n \in X$ with $d_X(p_n, q_n) < 1/n$ and $d_Y(f(p_n), f(q_n)) \geq \epsilon$. In this way we can construct sequences (p_n) and (q_n) as suggested in the question.

Rather than use Theorem 2.37, let's use the slightly stronger Theorem 3.6a); one might regard 3.6a) as a corollary of 2.37. Let (p_{n_k}) be a convergent subsequence of (p_n) and let $(q_{n_{k_l}})$ be a convergent subseqence of (q_{n_k}). Now $(p_{n_{k_l}})$ and $(q_{n_{k_l}})$ are convergent and still have the property stated in the question. Thus, we may suppose without loss of generality that (p_n) and (q_n) are convergent.

Write p for the limit of (p_n) by p and q for the limit of (q_n). Then

$$d(p,q) \leq d(p,p_n) + d(p_n,q_n) + d(q_n,q) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Thus $p = q$. f is continuous at p and so there exists a $\delta > 0$ such that

$$d_X(x,p) < \delta \implies d_Y(f(x), f(p)) < \epsilon/2.$$

Choose $n \in \mathbb{N}$ such that $d_X(p_n, p) < \delta$ and $d_X(q_n, p) < \delta$. Then

$$d(f(p_n), f(q_n)) \leq d(f(p_n), f(p)) + d(f(q_n), f(p)) < \epsilon,$$

a contradiction. We conclude that f is uniformly continuous.

3

Let $f : I \longrightarrow I$ be a continuous function. Define $g : I \longrightarrow \mathbb{R}$ by $g(x) = x - f(x)$. Then

$$g(0) = 0 - f(0) \leq 0 \quad \text{and} \quad 0 \leq 1 - f(1) = g(1).$$

If equality holds in either case then we are done. Otherwise $g(0) < 0 < g(1)$. One can check that g is continuous (use Theorem 4.6 and 4.4a)) and so we can apply Theorem 4.23 to find a point $x \in (0,1)$ such that $g(x) = 0$, i.e. $f(x) = x$.

4

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and suppose that $\lim_{x\longrightarrow+\infty} f(x)$ and $\lim_{x\longrightarrow-\infty} f(x)$ exist and are finite. Let A_+ and A_- be their respective values. We will show f is uniformly continuous.

Let $\epsilon > 0$. Choose $K_+ \in \mathbb{N}$ such that

$$x \geq K_+ - 1 \implies |f(x) - A_+| < \epsilon/2.$$

Choose $K_- \in \mathbb{N}$ such that

$$-x \geq K_- - 1 \implies |f(x) - A_-| < \epsilon/2.$$

Let $K = \max\{K_+, K_-\}$. $X = [-K-1, K+1]$ is compact and so by Theorem 4.19, $f|_X$ is uniformly continuous. Choose $\delta \in (0,1)$ such that

$$|x|, |x'| \leq K + 1 \quad \text{and} \quad |x - x'| < \delta \implies |f(x) - f(x')| < \epsilon.$$

We claim that

$$|x - x'| < \delta \implies |f(x) - f(x')| < \epsilon.$$

If this is true then f is uniformly continuous and we are done.

Suppose $|x - x'| < \delta$. Because $\delta < 1$, there are three cases to consider:

1. $|x|, |x'| \leq K + 1$ in which case $|f(x) - f(x')| < \epsilon$.
2. $x, x' \geq K - 1 \geq K_+ - 1$ in which case
$$|f(x) - f(x')| \leq |f(x) - A_+| + |f(x') - A_+| < \epsilon.$$
3. $-x, -x' \geq K - 1 \geq K_- - 1$ in which case
$$|f(x) - f(x')| \leq |f(x) - A_-| + |f(x') - A_-| < \epsilon.$$

5

Let K be a compact metric space with metric d and suppose $f : K \longrightarrow K$ is distance preserving, meaning that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in K$.

Suppose for contradiction that $K \neq f(K)$. Then we can choose $p_0 \in K$ with $p_0 \notin f(K)$. Inductively let $p_n = f(p_{n-1})$ for $n \in \mathbb{N}$. By Theorem 3.6a), $(p_n)_{n=1}^{\infty}$ has a convergent subsequence $(p_{n(k)})_{k=1}^{\infty}$.

Fix $r \in \mathbb{N}$. Since $(p_{n(k)})$ is Cauchy there exists $N_r \in \mathbb{N}$ such that

$$k \geq l \geq N_r \implies d(p_{n(k)}, p_{n(l)}) < 1/r.$$

Using that f is distance preserving and our inductive definition of (p_n) we see that

$$k > l \geq N_r \implies d(p_{n(k)-n(l)}, p) < 1/r.$$

Suppose inductively that we have $1 \leq m_1 < \ldots < m_{r-1}$ such that $d(p_{m_s}, p) < 1/s$ for all $s < r$. Let $m_r = n(N_r + m_{r-1} + 1) - n(N_r)$. Then $m_r > m_{r-1}$ and $d(p_{m_r}, p) < 1/r$. In this way we may construct a subsequence of $(p_n)_{n=1}^{\infty}$ converging to p_0.

f is easily seen to be continous and so $f(K)$ is compact. Thus $f(K)$ is closed and since $p_n \in f(K)$ for each $n \in \mathbb{N}$ we obtain $p \in f(K)$, a contradiction.

6

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and suppose that

$$|f(x) - f(y)| \leq (x - y)^2 \quad \text{for all} \ \ x, y \in \mathbb{R}.$$

Fix $x \in \mathbb{R}$. Then for $y \neq x$,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y| \longrightarrow 0 \quad \text{as} \quad y \longrightarrow x.$$

Thus $f'(x) = 0$. By Theorem 5.11b), f is constant.