Problem Set 6, 18.100B/C, Fall 2011

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1

a

Let $a_n = n^3 z^n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{n^3}|z| \geq |z|$$

and

$$\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow |z| \quad \text{as} \quad n \longrightarrow \infty.$$

One sees using Theorem 3.34 that $\sum n^3 z^n$ converges when $|z| < 1$ and diverges when $|z| \geq 1$ and the radius of convergence is 1.

b

Let $a_n = (2^n/n!)z^n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{n+1}|z| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

By Theorem 3.34a) $\sum(2^n/n!)z^n$ always converges so that the radius of convergence is ∞.

c

Let $a_n = 2^n/n^2$. Then

$$\sqrt[n]{|a_n|} = 2n^{-2/n} \longrightarrow 2 \quad \text{as} \quad n \longrightarrow \infty,$$

by Theorem 3.20c) and Theorem 3.3c) and d). By Theorem 3.39 the radius of convergence is 1/2.

d

Let $a_n = n^3/3^n$. Then

$$\sqrt[n]{|a_n|} = 3^{-1}n^{3/n} \longrightarrow 1/3 \quad \text{as} \quad n \longrightarrow \infty,$$

by Theorem 3.20c) and Theorem 3.3c) and d). By Theorem 3.39 the radius of convergence is 3.

2

This question says that given any "sublinear sequence" (a_n), the sequence (a_n/n) converges. We also have an explicit description for the limit.

a

Let (a_n) be a sequence such that $a_{n+m} \leq a_n + a_m$ for all $n, m \in \mathbb{N}$ and let $I = \inf\{\frac{a_n}{n} : n \in \mathbb{N}\} \in \mathbb{R} \cup \{-\infty\}$. Set $a_0 = 0$ for convenience. There are two cases.

1. $I \in \mathbb{R}$.

   Given $\epsilon > 0$, $I + \epsilon$ is not a lower bound for $\{\frac{a_n}{n} : n \in \mathbb{N}\}$ and so we can find an $l \in \mathbb{N}$ such that
   $$I \leq \frac{a_l}{l} < I + \frac{\epsilon}{2}.$$
   Choose $N \in \mathbb{N}$ such that $\max\{a_1, \dots, a_{l-1}\}/N < \epsilon/2$. Let $n \geq N$. By the division algorithm we can find $m, r \in \mathbb{N} \cup \{0\}$ such that
   $$r < l \quad \text{and} \quad n = ml + r$$
   Now $a_n \leq ma_l + a_r$ and so
   $$I \leq \frac{a_n}{n} \leq \frac{ma_l}{ml} + \frac{a_r}{N} < \left(I + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2} = I + \epsilon$$
   and so (a_n/n) converges to I.

2. $I = -\infty$.

   Let $K \in \mathbb{N}$. We can find an $l \in \mathbb{N}$ such that
   $$\frac{a_l}{l} < -(K+1).$$
   Choose $N \in \mathbb{N}$ such that $\max\{a_1, \dots, a_{l-1}\}/N < 1$. Let $n \geq N$. By the division algorithm we can find $m, r \in \mathbb{N} \cup \{0\}$ such that
   $$r < l \quad \text{and} \quad n = ml + r$$
   Now $a_n \leq ma_l + a_r$ and so
   $$\frac{a_n}{n} \leq \frac{a_l}{l} + \frac{a_r}{N} < -(K+1) + 1 = -K$$
   and so (a_n/n) tends to $I = -\infty$.

b

If you need convincing that lim sup and lim inf are useful concepts then here is another argument.

We have functions $m, r : \mathbb{N}^2 \longrightarrow \mathbb{N} \cup \{0\}$ defined by the following properties:

$$n = m(n,l)l + r(n,l) \quad \text{and} \quad r(n,l) < l.$$

For any $n \in \mathbb{N}$ we obtain

$$\frac{a_n}{n} \leq \frac{a_l}{l} + \frac{a_{r(n,l)}}{n}$$

as above. Fix $l \in \mathbb{N}$. Because $0 \leq r(n,l) < l$ we obtain

$$\limsup_{n \longrightarrow \infty} \frac{a_n}{n} \leq \frac{a_l}{l} \implies \limsup_{n \longrightarrow \infty} \frac{a_n}{n} \leq \inf \left\{ \frac{a_l}{l} : l \in \mathbb{N} \right\}.$$

Clearly,

$$\liminf_{n \longrightarrow \infty} \frac{a_n}{n} \geq \inf \left\{ \frac{a_l}{l} : l \in \mathbb{N} \right\}$$

and so

$$\lim_{n \longrightarrow \infty} \frac{a_n}{n} = \inf \left\{ \frac{a_l}{l} : l \in \mathbb{N} \right\}.$$

3

1. First we show that (a_n) is bounded:

   By assumption each $a_n \geq 0$. Also, for $n \in \mathbb{N}$

   $$a_{n+1} - a_1 = \sum_{k=1}^{n} (a_{k+1} - a_k) \leq \sum_{k=1}^{n} \frac{1}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

   so that for all $n \in \mathbb{N}$

   $$a_n \leq a_1 + \frac{\pi^2}{6}.$$

2. Thus (a_n) has a convergent subsequence. Let (a_{n_k}) be such a subsequence and suppose that it converges to a.

3. We show that (a_n) converges to a:

   Let $\epsilon > 0$. Choose $K \in \mathbb{N}$ such that

   $$k, l \geq K \implies |a_{n_k} - a_{n_l}| < \frac{\epsilon}{4} \quad \text{and} \quad \sum_{j=n_K}^{\infty} \frac{1}{j^2} < \frac{\epsilon}{4}.$$

   Suppose $l \geq n_K$. Then

   $$a_{l+1} - a_{n_K} = \sum_{j=n_K}^{l} (a_{j+1} - a_j) \leq \sum_{j=n_K}^{l} \frac{1}{j^2} < \frac{\epsilon}{4} < \frac{\epsilon}{2}.$$

   Also, we can choose a $k \geq K$ with $n_k > l$. Then

   $$a_{n_K} - a_l = (a_{n_K} - a_{n_k}) + \sum_{j=l}^{n_k - 1} (a_{j+1} - a_j) \leq (a_{n_K} - a_{n_k}) + \sum_{j=l}^{n_k - 1} \frac{1}{j^2} < \frac{\epsilon}{2}.$$

   Thus

   $$l \geq n_K \implies |a_l - a_{n_K}| < \frac{\epsilon}{2}$$

   and so

   $$l \geq n_K \implies |a_l - a| \leq |a_l - a_{n_K}| + |a_{n_K} - a| \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon.$$

Alternatively, there is a cheap trick one can use.

Let $b_1 = a_2 + 2$ and $b_n = a_n + \frac{1}{n-1}$ for $n > 1$. Then $b_2 < b_1$ and for $n > 1$,

$$b_{n+1} = a_{n+1} + \frac{1}{n} \leq \left(a_n + \frac{1}{n^2}\right) + \frac{1}{n} < a_n + \frac{1}{n-1} = b_n$$

since

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

Thus (b_n) is bounded below by 0 and monotone decreasing and so (b_n) is convergent. Since $(1/n)$ is convergent, so is (a_n).

4

Let (a_n) be a sequence and suppose that $\sum |a_{n+1} - a_n|$. Since absolute convergence implies convergence (Theorem 3.45), $\sum (a_{n+1} - a_n)$ converges. This means that $(a_n - a_1)_{n=1}^{\infty}$ converges and so (a_n) converges by Theorem 3.3b).

The converse is untrue. Let $a_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}$. By remark 3.46, (a_n) converges. However, remark 3.46 also tells us that $\sum_{n=1}^{\infty} 1/n$ does not converge. Since $|a_{n+1} - a_n| = 1/(n+1)$ we have the required counterexample.

5

If (a_n) converges, then (a_{n+1}) converges and so by Theorem 3.3a)b), $(2a_{n+1} - a_n)$ converges.

Suppose that (a_n) is sequence and that $(2a_{n+1} - a_n)$ converges to b. We will show that (a_n) converges to b. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |(2a_{n+1} - a_n) - b| < \epsilon.$$

Thus

$$\begin{aligned} n \geq N &\implies 2|a_{n+1} - b| \leq |2(a_{n+1} - b) - (a_n - b)| + |a_n - b| < \epsilon + |a_n - b| && (1) \\ &\implies |a_{n+1} - b| < (\epsilon + |a_n - b|)/2 && (2) \end{aligned}$$

Choose $M \in \mathbb{N}$ such that

$$M > N \quad \text{and} \quad 2^{N-M}|a_N - b| < \epsilon$$

Then using (2) inductively, we see that

$$n \geq M \implies |a_n - b| < \epsilon \sum_{j=1}^{n-N} 2^{-j} + 2^{N-n}|a_N - b| \leq \epsilon \sum_{j=1}^{\infty} 2^{-j} + 2^{N-M}|a_N - b| < 2\epsilon$$

and we are done.

6

a

Let $f : X \longrightarrow Y$ be a continuous map between metric spaces. Suppose E is dense in X. We will show that $f(E)$ is dense in $f(X)$.

Let $y \in f(X)$ and $\epsilon > 0$. We must show there exists an $x' \in E$ with $d_Y(y, f(x')) < \epsilon$. By definition of $f(X)$ there exists an $x \in X$ such that $f(x) = y$ and because f is continuous at x there exists a $\delta > 0$ such that

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \epsilon$$

Because E is dense in X we can find an $x' \in E$ such that $d_X(x, x') < \delta$. Then $d_Y(y, f(x')) < \epsilon$.

b

Let $f, g : X \longrightarrow Y$ be continuous maps between metric spaces. Suppose E is dense in X and that $f|E = g|E$. We will show that $f = g$.

Let $x \in X$ and $\epsilon > 0$. Since f and g are continuous at x there exists a $\delta > 0$ such that

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')),\ d_Y(g(x), g(x')) < \epsilon/2.$$

Since E is dense in X we may pick $x' \in E$ with $d_X(x, x') < \delta$. Then

$$d_Y(f(x), g(x)) \leq d_Y(f(x), f(x')) + d_Y(f(x'), g(x')) + d_Y(g(x'), g(x)) < \epsilon/2 + 0 + \epsilon/2 = \epsilon.$$

Since this holds for all $\epsilon > 0$ we must have $d_Y(f(x), g(x)) = 0$ and thus $f(x) = g(x)$. Since x was arbitary we are done.

7

We will prove the last part first.

a

Suppose $f : (a, b) \longrightarrow \mathbb{R}$ is a real-valued convex function and that $a < s < t < u < b$. Let $\lambda = \frac{u-t}{u-s}$. Then $0 < \lambda < 1$ and $t = \lambda s + (1 - \lambda)u$. Thus

$$f(t) \leq \lambda f(s) + (1 - \lambda) f(u).$$

Now

$$\begin{aligned} \frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} &\iff (u - s)(f(t) - f(s)) \leq (t - s)(f(u) - f(s)) \\ &\iff (u - s) f(t) \leq (u - t) f(s) + (t - s) f(u) \\ &\iff f(t) \leq \lambda f(s) + (1 - \lambda) f(u). \end{aligned}$$

Similarly, we see that

$$\frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

b

Fix $x \in (a,b)$ and $\delta > 0$ such that $[x-\delta, x+\delta] \subset (a,b)$. Let

$$C_- = \frac{f(x) - f(x-\delta)}{\delta} \quad \text{and} \quad C_+ = \frac{f(x+\delta) - f(x)}{\delta}.$$

For $z \in (x, x+\delta)$

$$\begin{aligned} a < x - \delta < x < z < x + \delta < b &\implies C_- \leq \frac{f(z) - f(x)}{z - x} \leq C_+ \\ &\implies C_-|z-x| \leq f(z) - f(x) \leq C_+|z-x| \end{aligned}$$

and for $z \in (x-\delta, x)$

$$\begin{aligned} a < x - \delta < z < x < x + \delta < b &\implies C_- \leq \frac{f(x) - f(z)}{x - z} \leq C_+ \\ &\implies C_-|z-x| \leq f(x) - f(z) \leq C_+|z-x| \end{aligned}$$

Hence letting $C = \max\{|C_+|, |C_-|\}$ we have

$$z \in (x-\delta, x+\delta) \implies |f(z) - f(x)| \leq C|z-x|$$

which shows continuity at x.

c

Let $f : (a,b) \longrightarrow \mathbb{R}$ be a convex function and let $g : (c,d) \longrightarrow \mathbb{R}$ be an increasing convex function. Suppose that $f(a,b) \subset (c,d)$ so that it makes sense to consider the composite

$$h : (a,b) \longrightarrow \mathbb{R}, \;\; x \longrightarrow g(f(x)).$$

We will prove that h is convex.

Suppose that $x, y \in (a,b)$ and $\lambda \in (0,1)$. Because f is convex

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y);$$

because g is increasing and convex

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &\leq g(\lambda f(x) + (1-\lambda) f(y)) \\ &\leq \lambda h(x) + (1-\lambda) h(y). \end{aligned}$$