Problem Set 5, 18.100B/C, Fall 2011

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October 18, 2011

1

a

Lemma: Let (a_n) be a bounded sequence of real numbers,

$$E = \{a \in \mathbb{R} : \text{ there exists a subsequence } (a_{n_k}) \text{ of } (a_n) \text{ with } a_{n_k} \longrightarrow a\}$$

and $b_n = \sup\{a_m : m \geq n\}$. Then $\sup E = \inf b_n$.

Proof: Write b for $\inf b_n$. It is enough to show b is an upper bound for E lying in E.

1. b is an upper bound for E.

   Let $a \in E$. Then there exists a subsequence (a_{n_k}) of (a_n) converging to a. Because $a_n \leq b_n$ for all $\in \mathbb{N}$ and (b_n) is a monotone non-increasing sequence we have

   $$k \geq r \implies n_k \geq r \implies a_{n_k} \leq b_{n_k} \leq b_r.$$

   Thus $a \leq b_r$ for all r and so $a \leq b$.

2. $b \in E$.

   For each $s \in \mathbb{N}$ there exists an $r(s) \geq s$ with

   $$b_s - 1/s \leq a_{r(s)} \leq b_s.$$

   Define an increasing sequence of natural numbers by letting $n_1 = r(1)$ and $n_k = r(n_{k-1} + 1)$. Then

   $$b_{n_{k-1}+1} - 1/(n_{k-1} + 1) \leq a_{n_k} \leq b_{n_{k-1}+1}$$

   Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > 1/\epsilon$ and

   $$n \geq N \implies 0 \leq b_n - b < \epsilon.$$

   Then

   $$k \geq N \implies |a_{n_k} - b| < \epsilon.$$

   Thus a_{n_k} converges to b, which implies $b \in E$.

This lemma answers the question since

$$\limsup_{n\longrightarrow\infty} a_n = \sup E$$

by definition and

$$\lim_{n\longrightarrow\infty}(\sup\{a_m : m \geq n\}) = \lim_{n\longrightarrow\infty} b_n = \inf b_n$$

because b_n is monotone non-increasing.

b

i

Let (a_n) and (b_n) be bounded sequences of real numbers. Then $(a_n + b_n)$ is a bounded sequence. In the proof of the lemma in part (a) we showed that we can find a subsequence $(a_{n_k} + b_{n_k})$ of $(a_n + b_n)$ which converges to

$$\limsup_{n\longrightarrow\infty}(a_n + b_n).$$

By passing to a finer subsequence if necessary we can assume that (a_{n_k}) and (b_{n_k}) are convergent. Then

$$\limsup_{n\longrightarrow\infty}(a_n + b_n) = \lim_{k\longrightarrow\infty}(a_{n_k} + b_{n_k}) = \lim_{k\longrightarrow\infty} a_{n_k} + \lim_{k\longrightarrow\infty} b_{n_k} \leq \limsup_{n\longrightarrow\infty} a_n + \limsup_{n\longrightarrow\infty} b_n$$

The second equality uses Theorem 3.3(a) and the final inequality uses the definition of $\limsup$.

ii

Suppose that (a_n) convergent. Choose a subsquence (b_{n_k}) of (b_n) which converges to $\limsup_{n\longrightarrow\infty} b_n$. Then

$$\limsup_{n\longrightarrow\infty} a_n + \limsup_{n\longrightarrow\infty} b_n = \lim_{k\longrightarrow\infty} a_{n_k} + \lim_{k\longrightarrow\infty} b_{n_k} = \lim_{k\longrightarrow\infty}(a_{n_k} + b_{n_k}) \leq \limsup_{n\longrightarrow\infty}(a_n + b_n)$$

The first equality uses the fact that $\limsup_{n\longrightarrow\infty} a_n = \lim_{k\longrightarrow\infty} a_{n_k} = \lim_{n\longrightarrow\infty} a_n$ for convergent sequences, the second equality uses Theorem 3.3(a) and the final inequality follows by definition.

iii

Let $a_{2n-1} = b_{2n} = 1$ and $a_{2n} = b_{2n-1} = -1$. Then

$$\limsup_{n\longrightarrow\infty} a_n = \limsup_{n\longrightarrow\infty} b_n = 1 \quad \text{and} \quad \limsup_{n\longrightarrow\infty}(a_n + b_n) = 0$$

so that the equality in (i) is strict.

2

a

Lemma: Suppose (a_n), (b_n) and (c_n) are sequences of real numbers such that (a_n) and (c_n) converge to l and

$$a_n \leq b_n \leq c_n \text{ for each } n \in \mathbb{N}.$$

Then (b_n) converges to l.

Proof: Let $\epsilon > 0$. We can choose $N', N'' \in \mathbb{N}$ such that

$$n \geq N' \implies |a_n - l| < \epsilon/3 \quad \text{and} \quad n \geq N'' \implies |c_n - l| < \epsilon/3$$

Let $N = \max\{N', N''\}$. Then

$$n \geq N \implies 0 \leq b_n - a_n \leq c_n - a_n \leq |c_n - l| + |a_n - l| < 2\epsilon/3$$

so

$$n \geq N \implies |b_n - l| \leq (b_n - a_n) + |a_n - l| < \epsilon.$$

b

Let $k \in \mathbb{N}$ and let $x_1 \geq x_2 \geq \ldots \geq x_k \geq 0$. For each $n \in \mathbb{N}$

$$\max\{x_i\}^n \leq x_1^n + \ldots + x_k^n \leq k \max\{x_i\}^n$$

so that

$$\max\{x_i\} \leq (x_1^n + \ldots + x_k^n)^{1/n} \leq \sqrt[n]{k} \max\{x_i\}.$$

Let $a_n = \max\{x_i\}$, $b_n = (x_1^n + \ldots + x_k^n)^{1/n}$ and $c_n = \sqrt[n]{k} \max\{x_i\}$. By Theorem 3.3($b$) and 3.20($b$), (c_n) converges to $\max\{x_i\}$. By part a) so does (b_n).

3

The only if direction is easy so we'll do the if direction.

Let X be a metric space. Suppose $x \in X$ and (x_n) is a sequence in X and that (x_n) does not converge to x. Then there exists an $\epsilon > 0$ with the following property: for each $s \in \mathbb{N}$, there exists an $r(s) \in \mathbb{N}$ with $r(s) \geq s$ and $d(x_{r(s)}, x) \geq \epsilon$. Inductively define $n_1 = r(1)$ and $n_k = r(n_{k-1} + 1)$. Then (x_{n_k}) is a subsequence with $d(x_{n_k}, x) \geq \epsilon$ for all $k \in \mathbb{N}$. Hence, no subsequence of (x_{n_k}) converges to x and we are done.

4

a

Reflexivity and symmetry are obvious since $d(x, x) = 0$ and $d(x, y) = d(y, x)$.

Suppose (p_n) and (q_n) are equivalent and (q_n) and (r_n) are equivalent. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that

$$n \geq N \implies d(p_n, q_n) < \epsilon/2, \ \ d(q_n, r_n) < \epsilon/2$$

Then

$$n \geq N \implies d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n) < \epsilon,$$

which shows (p_n) and (r_n) are equivalent and the relation is transitive.

b

Let (p_n), (p'_n) and (q_n), (q'_n) be equivalent Cauchy sequences. The triangle inequality gives

$$|d(p_n, q_n) - d(p'_n, q'_n)| \leq d(p_n, p'_n) + d(q_n, q'_n).$$

Suppose $(d(p_n, q_n))$ converges to $d \in \mathbb{R}$. Given $\epsilon > 0$ choose $N \in \mathbb{N}$ such that

$$n \geq N \implies |d(p_n, q_n) - d|, \ d(p_n, p'_n), \ d(q_n, q'_n) < \epsilon/3$$

Then

$$n \geq N \implies |d(p'_n, q'_n) - d| < \epsilon.$$

Thus the definition of $\Delta(P, Q)$ makes sense.

Clearly, $\Delta(P, P) = 0$ and $\Delta(P, Q) = \Delta(Q, P)$. Suppose $P, Q, R \in X^*$ and $(p_n) \in P$, $(q_n) \in Q$, $(r_n) \in R$. Taking limits in

$$d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n)$$

gives $\Delta(P, R) \leq \Delta(P, Q) + \Delta(Q, R)$.

Suppose $\Delta(P, Q) = 0$. Then $\lim_{n \longrightarrow \infty} d(p_n, q_n) = 0$ which implies (p_n) and (q_n) are equivalent and hence $P = Q$.

c

Let $(P_n)_{n=1}^{\infty}$ be a Cauchy sequence in X^*. Let $(p_r^n)_{r=1}^{\infty} \in P_n$.

For each $n \in \mathbb{N}$, $(p_r^n)_{r=1}^{\infty}$ is Cauchy and so

$$\exists R_n \in \mathbb{N} \ : \ r, s \geq R_n \implies d(p_r^n, p_s^n) < 1/n.$$

Because the sequence $(P_n)_{n=1}^{\infty}$ is Cauchy, for each $k \in \mathbb{N}$

$$\exists N_k \in \mathbb{N} \ : \ n, m \geq N_k \implies \lim_{r \longrightarrow \infty} d(p_r^n, p_r^m) < 1/k.$$

By replacing N_k by $\max\{N_1, \dots N_k, k\}$ if necessary we may assume that $N_1 \leq N_2 \leq N_3 \leq N_4 \leq \dots$ and $N_k \geq k$.

Let $p_k = p_{R_{N_k}}^{N_k}$. We now set about showing that $(p_k)_{k=1}^{\infty}$ is Cauchy. Let $k \geq l$.

$$k \geq l \implies N_k \geq N_l \implies \lim_{r \longrightarrow \infty} d(p_r^{N_k}, p_r^{N_l}) < 1/N_l.$$

and so we may choose $r \geq \max\{R_{N_k}, R_{N_l}\}$ such that $d(p_r^{N_k}, p_r^{N_l}) < 1/N_l$. Then

$$\begin{aligned} d(p_k, p_l) = d\left(p^{N_k}_{R_{N_k}}, p^{N_l}_{R_{N_l}}\right) &\leq d\left(p^{N_k}_{R_{N_k}}, p_r^{N_k}\right) + d\left(p_r^{N_k}, p_r^{N_l}\right) + d\left(p_r^{N_l}, p^{N_l}_{R_{N_l}}\right) \\ &< 1/N_k + 1/N_l + 1/N_l \\ &\leq 3/l \end{aligned}$$

which shows $(p_k)_{k=1}^{\infty}$ is Cauchy. Thus $(p_k)_{k=1}^{\infty}$ defines an element $P \in X^*$. We now set about showing that $(P_n)_{n=1}^{\infty}$ converges to P. Note that

$$\Delta(P, P_n) = \lim_{r \longrightarrow \infty} d(p_r, p_r^n) \quad \text{and} \quad d(p_r, p_r^n) = d\left(p^{N_r}_{R_{n_r}}, p_r^n\right).$$

Let $k \in \mathbb{N}$ and fix $n \geq N_k$. Let $r \geq \max\{k, R_n\}$.

$$N_r, n \geq N_k \implies \lim_{s \longrightarrow \infty} d\left(p_s^{N_r}, p_s^n\right) < 1/k$$

and so we may choose $s \geq R_{N_r}, R_n$ such that $d\left(p_s^{N_r}, p_s^n\right) < 1/k$. Then

$$\begin{aligned} d(p_r, p_r^n) = d\left(p^{N_r}_{R_{n_r}}, p_r^n\right) &\leq d\left(p^{N_r}_{R_{n_r}}, p_s^{N_r}\right) + d\left(p_s^{N_r}, p_s^n\right) + d\left(p_s^n, p_r^n\right) \\ &< 1/N_r + 1/k + 1/n \\ &\leq 1/r + 1/k + 1/N_k \\ &\leq 3/k \end{aligned}$$

Thus $\Delta(P, P_n) \leq 3/k$ whenever $n \geq N_k$ which shows $(P_n)_{n=1}^{\infty}$ converges to P. Thus X^* is complete.

d

Trivial.

e

Let $P \in X^*$ and $(p_n) \in P$. Consider the sequence $(\varphi(p_n))$ in X^*. Let $\epsilon > 0$. Since (p_n) is Cauchy there exists an $N \in \mathbb{N}$ such that

$$n, m \geq N \implies d(p_n, p_m) < \epsilon/2.$$

Letting m tend to infinity we obtain

$$n \geq N \implies \Delta(\varphi(p_n), P) \leq \epsilon/2 < \epsilon.$$

Thus $(\varphi(p_n))$ converges to P in X^* which shows $\varphi(X)$ is dense in X^*.

Suppose in addition that X is complete. Then (p_n) converges to some $p \in X$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies d(p_n, p) < \epsilon.$$

Thus

$$n \geq N \implies \Delta(\varphi(p_n), \varphi(p)) < \epsilon.$$

This means $\varphi(p_n)$ converges to $\varphi(p)$ in X^*. Thus $\varphi(p) = P$ and we see $\varphi(X) = X^*$.

5

a

$$\sum_{n=1}^{k} a_n = \sqrt{k+1} - 1$$

which diverges.

b

$$|a_n| = \frac{\sqrt{n+1} - \sqrt{n}}{n} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} < \frac{1}{n\sqrt{n}}$$

$\sum \frac{1}{n\sqrt{n}}$ converges by Theorem 3.28 and so $\sum a_n$ converges by Theorem 3.25(a).

c

$$\lim_{n \longrightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \longrightarrow \infty} (\sqrt[n]{n} - 1) = 0$$

by Theorem 3.20(c) and Theorem 3.3(a). Thus, by Theorem 3.3(a), $\sum a_n$ converges.

d

i

Suppose $|z| > 1$. Then

$$\frac{1}{|z|^n + 1} \leq |a_n| \leq \frac{1}{|z|^n - 1}$$

Because

$$\left(\sqrt[n]{|z|^n + 1}\right)^{-1}, \left(\sqrt[n]{|z|^n - 1}\right)^{-1} \longrightarrow |z|^{-1} \text{ as } n \longrightarrow \infty$$

we see (use question 2(a)) that

$$\lim_{n \longrightarrow \infty} \sqrt[n]{|a_n|} = |z|^{-1} < 1$$

and Theorem 3.3(a) tells us $\sum a_n$ converges.

ii

Suppose $|z| \leq 1$. Then $|1 + z^n| \leq 2$ so that

$$|a_n| \geq 1/2.$$

The series cannot converge since the individual terms do not tend to zero.

6

Let (a_n) be a sequence of positive numbers which tends to 0 but such that $\sum a_n$ diverges. For $n \in \mathbb{N}$ let

$$A_n = \sum_{k=1}^{n} a_k.$$

For $n \in \mathbb{N}$, let $b_{n+1} = \sqrt{A_{n+1}} - \sqrt{A_n}$ and let $b_1 = \sqrt{A_1}$. Then

$$\frac{b_{n+1}}{a_{n+1}} = \frac{\sqrt{A_{n+1}} - \sqrt{A_n}}{a_{n+1}} = \frac{A_{n+1} - A_n}{a_{n+1}\left(\sqrt{A_{n+1}} + \sqrt{A_n}\right)} = \frac{1}{\sqrt{A_{n+1}} + \sqrt{A_n}} \leq \frac{1}{\sqrt{A_n}}$$

converges to 0 because $\sqrt{A_n}$ tends to infinity. However,

$$\sum_{k=1}^{n} b_k = \sqrt{A_n}$$

diverges.