Problem Set 4, 18.100B/C, Fall 2011

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A sketch solution for question 13 of the review sheet

Let X be a compact metric space and $K_1 \supset K_2 \supset K_3 \supset \ldots$ be a sequence of nested closed connected subsets.

Suppose for contradiction that there exist nonempty closed subsets A, B of $K = \bigcap_{i=1}^{\infty} K_i$ such that $A \cap B = \emptyset$ and $A \cup B = K$. Then A and B are closed in X. Thus we can choose open sets in X such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. (This is because compact Hausdorff spaces are normal - see Munkres.)

Consider $W_n = K_n \setminus (U \cup V)$. This set is closed and it is nonempty (since otherwise writing $(U \cap K_n) \cup (V \cap K_n) = K_n$ we see K_n is disconnected, a contradiction). Also, $W_n \supset W_{n+1}$ and so, because X is compact, $\bigcap_{i=1}^{\infty} W_n \neq \emptyset$. This gives the required contradiction since we have $\bigcap_{i=1}^{\infty} W_n = K \setminus (U \cup V) = \emptyset$.

1

Lemma: Suppose (s_n) is a sequence in $\mathbb{C}$ converging to s. Then $(|s_n|)$ converges to $|s|$ in $\mathbb{C}$ (or $\mathbb{R}$).

Proof: Let $\epsilon > 0$. Because (s_n) converges to s, there exists an $N \in \mathbb{N}$ such that

$$n \geq N \implies |s_n - s| < \epsilon$$

Since $||s_n| - |s|| \leq |s_n - s|$

$$n \geq N \implies ||s_n| - |s|| < \epsilon$$

and we are done.

The converse is not true. Let $s_n = (-1)^n$. For each $n \in \mathbb{N}$, $|s_n| = 1$ and so $(|s_n|)$ converges to 1. If (s_n) were convergent then (s_{2n}) and (s_{2n+1}) would converge with the same limit. However, (s_{2n}) has limit 1 and (s_{2n+1}) has limit -1.

2

Let X be a complete metric space with metric d, and let $f : X \longrightarrow X$ be a contraction, meaning that there exists $\lambda < 1$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in X$. Then there is a unique point $x_0 \in X$ such that $f(x_0) = x_0$.

Proof:

1. Existence: Let $x_1 \in X$ be arbitary and inductively let $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$. We will prove that (x_n) is a Cauchy sequence. Suppose inductively that

$$d(x_{r+1}, x_r) \leq \lambda^{r-1} d(x_2, x_1).$$

Then

$$d(x_{r+2}, x_{r+1}) = d(f(x_{r+1}), f(x_r)) \leq \lambda d(x_{r+1}, x_r) \leq \lambda^r d(x_2, x_1)$$

so that the above equation holds for all $r \in \mathbb{N}$. For $m > n$, by repeated use of the triangle inequality

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n).$$

Hence,

$$d(x_m, x_n) \leq (\lambda^{m-2} + \ldots \lambda^{n-1}) d(x_2, x_1) = \frac{\lambda^{n-1}(1 - \lambda^{m-n})}{1 - \lambda} d(x_2, x_1) \leq \frac{\lambda^{n-1}}{1 - \lambda} d(x_2, x_1).$$

Let $\epsilon > 0$. By theorem 3.20(e) and 3.3(b), there exists an $N \in \mathbb{N}$ such that

$$n \geq N \implies \lambda^{n-1} d(x_2, x_1) < \epsilon(1 - \lambda)$$

and so

$$m, n \geq N \implies d(x_m, x_n) < \epsilon,$$

which shows (x_n) is Cauchy. Since X is complete (x_n) converges to some $x_0 \in X$. Given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies d(x_0, x_n) < \frac{\epsilon}{2}$$

and so

$$d(x_0, f(x_0)) \leq d(x_0, x_{N+1}) + d(f(x_N), f(x_0)) \leq d(x_0, x_{N+1}) + \lambda d(x_N, x_0) < \epsilon.$$

Since ϵ was arbitary, $d(x_0, f(x_0)) = 0$ giving $x_0 = f(x_0)$, as required.

2. Uniqueness: If $f(x_0) = x_0$ and $f(y_0) = y_0$ then

$$d(x_0, y_0) = d(f(x_0), f(y_0)) \leq \lambda d(x_0, y_0) \implies (1 - \lambda) d(x_0, y_0) \leq 0 \implies d(x_0, y_0) \leq 0.$$

Thus $d(x_0, y_0) = 0$ giving $x_0 = y_0$.

3

a

Lemma: Suppose (a_n) is a sequence of non-negative real numbers. If (a_n) converges to a, then $(\sqrt{a_n})$ converges to $\sqrt{a}$.

1. Case 1: $a = 0$.

   Let $\epsilon > 0$. Since (a_n) converges to 0 there exists an $N \in \mathbb{N}$ such that

   $$n \geq N \implies |a_n| < \epsilon^2.$$

   Then

   $$n \geq N \implies |\sqrt{a_n}| < \epsilon,$$

   which shows that $(\sqrt{a_n})$ converges to 0.

2. Case 2: $a > 0$ (since $[0, \infty)$ is closed we cannot have $a < 0$).

   Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that

   $$n \geq N \implies |a_n - a| < \epsilon\sqrt{a}.$$

   Then

   $$n \geq N \implies |\sqrt{a_n} - \sqrt{a}| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \leq \frac{|a_n - a|}{\sqrt{a}} < \epsilon$$

   which shows $(\sqrt{a_n})$ converges to $\sqrt{a}$.

b

We have the following identity

$$\begin{aligned}\left[\sqrt{n^2+n} - n\right] - \frac{1}{2} &= \frac{\left[\sqrt{n^2+n} - \left(n + \frac{1}{2}\right)\right]\left[\sqrt{n^2+n} + \left(n + \frac{1}{2}\right)\right]}{\sqrt{n^2+n} + \left(n + \frac{1}{2}\right)} \\ &= \frac{\left(n^2+n\right) - \left(n + \frac{1}{2}\right)^2}{\sqrt{n^2+n} + \left(n + \frac{1}{2}\right)} \\ &= \frac{-1}{4\left[\sqrt{n^2+n} + \left(n + \frac{1}{2}\right)\right]}\end{aligned}$$

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ with $N > 1/\epsilon$. Then

$$n \geq N \implies \left|\left[\sqrt{n^2+n} - n\right] - \frac{1}{2}\right| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

so that $\sqrt{n^2+n} - n \longrightarrow \frac{1}{2}$ as $n \longrightarrow \infty$.

4

Theorem: Let K be a compact metric space, and $\{G_\alpha\}_{\alpha \in A}$ an open cover of K. Then there exists an $\epsilon > 0$ with the following property:

$$\text{for every } x \in K \text{ there exists an } \alpha \in A \text{ such that } N_\epsilon(x) \subset G_\alpha.$$

Proof: We proceed by contradiction. Suppose that we cannot find such an ϵ.

For each $n \in \mathbb{N}$ there exists a point $x_n \in K$ such that $N_{1/n}(x) \not\subset G_\alpha$ for every $\alpha \in A$; otherwise we could take ϵ to be equal to $1/n$. By Theorem 3.6(a), (x_n) has a convergent subsequence: write (x_{n_r}) for the subsequence and x for its limit. Choose $\alpha_0 \in A$ such that $x \in G_{\alpha_0}$. Since G_{α_0} is open we can choose $m \in \mathbb{N}$ such that $N_{2/m}(x) \subset G_{\alpha_0}$. We can also find $R \in \mathbb{N}$ such that

$$r \geq R \implies x_{n_r} \in N_{1/m}(x).$$

Choose $r \geq R$ so that $n_r \geq m$ and let $s = n_r$. Then

$$N_{1/s}(x_s) \subset N_{2/m}(x)$$

since

$$d(x_s, y) < 1/s \implies d(x, y) \leq d(x, x_s) + d(x_s, y) < 1/m + 1/s \leq 2/m.$$

Then

$$N_{1/s}(x_s) \subset U_{\alpha_0},$$

contradicting the choice of x_s.