Problem Set 3, 18.100B/C, Fall 2011

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1

Let X be a metric space and $E \subset X$. Let $\mathrm{cl}(E)$ denote the closure of E and let $\mathrm{int}(E)$ denote the interior of E.

a

We proved on the last sheet (problem 2) that for any E, $\mathrm{int}(E)$ is open and that if E is open then $\mathrm{int}(E) = E$. Thus $\mathrm{int}(\mathrm{int}(E)) = \mathrm{int}(E)$.

Similarly, it is proved in Rudin (2.27) that for any E, $\mathrm{cl}(E)$ is closed and that if E is closed then $\mathrm{cl}(E) = E$. Thus $\mathrm{cl}(\mathrm{cl}(E)) = \mathrm{cl}(E)$.

b

Lemma: $E \subset F \subset X \implies \mathrm{int}(E) \subset \mathrm{int}(F)$ and $\mathrm{cl}(E) \subset \mathrm{cl}(F)$.

Proof: Suppose $E \subset F$; then $E \subset \mathrm{cl}(F)$ and by 2.27 (a), (c) of Rudin, $\mathrm{cl}(E) \subset \mathrm{cl}(F)$. Also, $E^c \supset F^c$ gives $\mathrm{cl}(E^c) \supset \mathrm{cl}(F^c)$. By what we proved in problem 2 of the last sheet this gives $(\mathrm{int}(E))^c \supset (\mathrm{int}(F))^c$ so that $\mathrm{int}(E) \subset \mathrm{int}(F)$.

Lemma: $\mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(E)))) = \mathrm{int}(\mathrm{cl}(E))$.

Proof:

$$\mathrm{int}(\mathrm{cl}(E)) \subset \mathrm{cl}(E) \implies \mathrm{cl}(\mathrm{int}(\mathrm{cl}(E))) \subset \mathrm{cl}(\mathrm{cl}(E)) = \mathrm{cl}(E) \implies \mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(E)))) \subset \mathrm{int}(\mathrm{cl}(E))$$

and

$$\mathrm{int}(\mathrm{cl}(E)) \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(E))) \implies \mathrm{int}(\mathrm{cl}(E)) = \mathrm{int}(\mathrm{int}(\mathrm{cl}(E))) \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(E)))).$$

Corollary: $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}(E)))) = \mathrm{cl}(\mathrm{int}(E))$.

Proof: We proved last time (problem 2) that $\mathrm{cl}(E^c) = (\mathrm{int}(E))^c$ and we also have

$$\mathrm{int}(E^c) = ((\mathrm{int}(E^c))^c)^c = (\mathrm{cl}((E^c)^c))^c = (\mathrm{cl}(E))^c.$$

Now

$$\operatorname{int}(\operatorname{cl}(\operatorname{int}(\operatorname{cl}(E^c)))) = \operatorname{int}(\operatorname{cl}(E^c))$$

gives

$$(\operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{int}(E)))))^c = (\operatorname{cl}(\operatorname{int}(E)))^c$$

and applying $(-)^c$ gives the result.

c

Consider E with some sequence of the cl and int operations applied to it. By (a) we can assume no operation is applied twice in a row, and by (b) we can assume the sequence of operations has length less than or equal to 3. Thus the only possibilites are:

$$E,\ \operatorname{cl}(E),\ \operatorname{int}(E),\ \operatorname{int}(\operatorname{cl}(E)),\ \operatorname{cl}(\operatorname{int}(E)),\ \operatorname{cl}(\operatorname{int}(\operatorname{cl}(E))),\ \operatorname{int}(\operatorname{cl}(\operatorname{int}(E))).$$

d

Let $E = (\mathbb{Q} \cap (-\infty, -1]) \cup \{0\} \cup [1, 2) \cup (2, \infty) \subset \mathbb{R}$. Then

$$\operatorname{cl}(E) = (-\infty, -1] \cup \{0\} \cup [1, \infty),\ \operatorname{int}(E) = (1, 2) \cup (2, \infty)$$
$$\operatorname{int}(\operatorname{cl}(E)) = (-\infty, -1) \cup (1, \infty),\ \operatorname{cl}(\operatorname{int}(E)) = [1, \infty)$$
$$\operatorname{cl}(\operatorname{int}(\operatorname{cl}(E))) = (-\infty, -1] \cup [1, \infty),\ \operatorname{int}(\operatorname{cl}(\operatorname{int}(E))) = (1, \infty).$$

2

Lemma: $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

Proof: Just use the density of $\mathbb{Q}$ in $\mathbb{R}$ for each coordinate.

Theorem: Let $n \in \mathbb{N}$ and let $S \in \mathbb{R}^n$ be a set such that every point in S is isolated. Then S is at most countable.

Proof: Fix $s \in S$. Since s is an isolated point, there exists an $\tilde{r}_s > 0$ such that $N_{\tilde{r}_s}(s) \cap S = \{s\}$; let $r_s = \tilde{r}_s/2$ and pick an element $t_s \in N_{r_s}(s) \cap \mathbb{Q}^n$. Doing this for each s defines a function

$$f : S \longrightarrow \mathbb{Q}^n,\ \ s \longmapsto t_s.$$

We now go about showing that f is injective; since $\mathbb{Q}^n$ is countable this will show S is at most countable.

Suppose $f(s) = f(\tilde{s})$ and let $t = f(s)$. Then $t = t_s = t_{\tilde{s}} \in N_{r_s}(s) \cap N_{r_{\tilde{s}}}(\tilde{s})$. Thus

$$d(s, \tilde{s}) \leq d(t, s) + d(t, \tilde{s}) < r_s + r_{\tilde{s}} \leq \max\{\tilde{r}_s, \tilde{r}_{\tilde{s}}\}$$

so either $s \in N_{\tilde{r}_{\tilde{s}}}(\tilde{s})$ or $\tilde{s} \in N_{\tilde{r}_s}(s)$. In either case we obtain $s = \tilde{s}$.

3

a

$$\{1, 2, 3\} \subset \mathbb{R}$$

Open: No
Closed: Yes
Compact: Yes
Interior: $\emptyset$
Limit points: $\emptyset$
Closure: $\{1, 2, 3\}$

b

$$[-1, 0) \cup (0, 1] \subset \mathbb{R}$$

Open: No
Closed: No
Compact: No
Interior: $(-1, 0) \cup (0, 1)$
Limit points: $[-1, 1]$
Closure: $[-1, 1]$

c

$$\mathbb{Q} \subset \mathbb{R}$$

Open: No
Closed: No
Compact: No
Interior: $\emptyset$
Limit points: $\mathbb{R}$
Closure: $\mathbb{R}$

d

$$\mathbb{R} \setminus \mathbb{Q} \subset \mathbb{R}$$

Open: No
Closed: No
Compact: No
Interior: $\emptyset$
Limit points: $\mathbb{R}$
Closure: $\mathbb{R}$

e

$$\{(x,y) \in \mathbb{R}^2 : y > 0\} \subset \mathbb{R}^2$$

Open: Yes
Closed: No
Compact: No
Interior: $\{(x,y) : y > 0\}$
Limit points: $\{(x,y) : y \geq 0\}$
Closure: $\{(x,y) : y \geq 0\}$

f

$$\{(x,y) \in \mathbb{R}^2 : x \in [-1,0) \cup (0,1]\} \subset \mathbb{R}^2$$

Open: No
Closed: No
Compact: No
Interior: $\{(x,y) : x \in (-1,0) \cup (0,1)\}$
Limit points: $\{(x,y) : x \in [-1,1]\}$
Closure: $\{(x,y) : x \in [-1,1]\}$

4

Let

$$X = \{1/n : n \in \mathbb{N}\}$$

and let

$$Y = \{0\} \cup \bigcup_{n \in \mathbb{N}} \left\{ \left(\frac{1}{n} - \frac{1}{n+1} \right) X + \frac{1}{n+1} \right\}.$$

Lemma: Y is compact.

Proof: Let $\{U_\alpha : \alpha \in A\}$ be an open cover of Y so that

$$Y \subset \bigcup_{\alpha \in A} U_\alpha.$$

In particular

$$0 \in \bigcup_{\alpha \in A} U_\alpha$$

so there is some $\alpha_0 \in A$ with $0 \in U_{\alpha_0}$. Since U_{α_0} is open, there exists an $r_0 > 0$ with $N_{r_0}(0) \subset U_{\alpha_0}$.

For each $n \in \mathbb{N}$ there exists an $\alpha_n \in A$ such that

$$1/n \in U_{\alpha_n}$$

and we can choose $r_n > 0$ so that

$$N_{r_n}(1/n) \subset U_{\alpha_n}$$

Consider the set

$$Z = N_{r_0}(0) \cup \bigcup_{n=1}^{\lfloor 1/r_0 \rfloor} N_{r_n}(1/n) \subset U_{\alpha_0} \cup \bigcup_{n=1}^{\lfloor 1/r_0 \rfloor} U_{\alpha_n}$$

One can check $Y \setminus Z$ is finite: you should do this. For each point $z \in Z \setminus Y$ choose α_z such that $z \in U_{\alpha_z}$. Then

$$\{U_{\alpha_0}\} \cup \{U_{\alpha_n} : n \in \{1, \ldots, \lfloor 1/r_0 \rfloor\}\} \cup \{U_{\alpha_z} : z \in Z \setminus Y\}$$

is the required subcover. Alternatively, one can use Heine-Borel. I feel the argument just given is the same in difficulty but perhaps it is longer to write down. Y is bounded and to show it is closed one can write the complement as a union of open intervals.

Lemma: The limit points of Y are $\{0\} \cup X$.

Proof: Any other point is isolated and one checks that these are indeed limit points.

$\{0\} \cup X$ is a countable set so we have answered the question.

5

Let K be a compact metric space and $\epsilon > 0$. Since K is compact, the open cover

$$\{N_{\epsilon/2}(x) : x \in K\}$$

has a finite subcover, i.e. there exist $x_1, \ldots, x_n \in K$ such that

$$K = \bigcup_{i=1}^{n} N_{\epsilon/2}(x_i)$$

Let $N = n + 1$. Now suppose we are give N distinct points $z_1, \ldots, z_N$ in K. At least two of them must lie in the same set of the cover and so there exists an $i \in \{1, \ldots n\}$ and two points $z_r \neq z_s$ with $d(z_r, x_i) < \epsilon/2$ and $d(z_s, x_i) < \epsilon/2$. Then

$$d(z_r, z_s) \leq d(z_r, x_i) + d(z_s, x_i) < \epsilon$$

and we're done.

6

Let K be a compact metric space and fix $n \in \mathbb{N}$. Since K is compact the open cover

$$\{N_{1/n}(x) : x \in K\}$$

has finite subcover, i.e. there exist $x_1^{(n)}, \ldots, x_{r_n}^{(n)}$ such that

$$K = \bigcup_{i=1}^{r_n} N_{1/n}(x_i^{(n)}).$$

Doing this for each $n \in \mathbb{N}$ we obtain a countable collection of finite sets

$$\left\{\{x_1^{(n)}, \ldots, x_{r_n}^{(n)}\} : n \in \mathbb{N}\right\}.$$

Let D be the union of all these sets. D is at most countable since it is a countable union of finite sets (see problem 6 of last problem set). D is dense and so we are done.

To see the final statement is true let $x \in K$ and $\epsilon > 0$. We can choose $n \in \mathbb{N}$ such that $1/n < \epsilon$ and there is an $i \in \{1, \ldots, r_n\}$ such that

$$x \in N_{1/n}(x_i^{(n)}).$$

Thus $x_i^{(n)} \in N_{1/n}(x) \subset N_\epsilon(x)$, which shows D is dense in K, as required.